Journal of Computational Applied Mechanics 2023, 54 (1): 19-35
DOI: 10.22059/JCAMECH.2022.351684.778

RESEARCH PAPER

The couple stress analysis of Timoshenko micro-beams based on new considerations

Hossein Golbakhshi, M.R. Dashtbayazi [*], Ali Reza Saidi

Department of Mechanical Engineering, Shahid Bahonar University of Kerman, Kerman, Iran,

Abstract

In this paper, a new approach is proposed for the couple stress analysis of micro-beams. As the main assumption, power series expansions are assumed for the axial displacement. The lateral and transverse displacements are adopted according to the classical beam theories. It is demonstrated that this consideration imposes a decisive constraint of skew-symmetry on the couple-stress tensor. So, in the case of micro-beams, there is no need for referring to the main arguments in modified couple stress theory (M-CST). This approach also allows for revising the conventional boundary conditions in couple stress analysis of micro-beams. For the special case of Timoshenko micro-beams, the axial displacement is approximated by a first-order polynomial and a new set of boundary conditions similar to the classical model is developed. Benchmark problems are then considered for demonstrating the advantages of the proposed model.

Keywords: size effect, microstructure, skew-symmetric couple-stress tensor, structural theories, variational principle.

1. Introduction

The beam-like elements have widespread use in modern micro- and nano-electromechanical systems [1]. A series of experiments [2, 3] showed that the mechanical behavior of thin beams in micron and sub-micron scales is strongly size-dependent and cannot be described by the classical elasticity theory. So, the higher-order theories with additional degrees of freedom and intrinsic length-related parameters are developed for the analysis of small-scale structures. Shariati et al. 2021 [4] developed a molecular dynamic analysis for calibration of length-related parameters in the non-classical theories.

The nonlocal elasticity theory [5] has been widely used investigating the size-dependent mechanical behavior of nano-beams [6, 7], nano-disks [8] and nano-plates [9]. So, the strain gradient theory [10] is also used for the analysis of small-scale structures [11-13]. In order to reduce the additional variables and the number of required length-related constants, the original couple stress theory [14-16] developed based on a general non-symmetric couple-stress tensor. In special case of linear isotropic materials, only two extra length-related constants were required in the original couple stress theory to capture the size effect of microstructures.

Yang et al. [17] proposed an artificial equilibrium equation for couple moments and developed the modified couple stress theory (M-CST) based on a symmetric couple-stress tensor and only one required length scale parameter. Therefore, M-CST has been widely used for the analysis of microstructures. (see e.g., [18-21]). However, the equilibrium of couple moments which is used to imply the symmetry of the couple-stress tensor is not deduced from any accepted principle in continuum mechanics [22]. Hadjesfandiari and Dargush [23] proposed an alternative consistent couple stress theory (C-CST) based on a skew-symmetric couple-stress tensor. Recently, C-CST has been widely used for the analysis of isotropic and FGM micro/nano-beams [24-27]. However, Münch, Neff et al. [28]

[*] Corresponding author. *E-mail address:* dashtbayazi@uk.ac.ir

raised serious doubts on validity of the main arguments proposed by [23].

In order to develop a simplified micropolar beam theory, Ramezani et al. [29] assumed power series expansions for both axial displacement and the non-zero micro-rotation. It is noted that in the couple stress theory, the rotation vector is completely dependent on the displacement field and any approximation considered for the displacements, may affect the characteristics of the rotational degree of freedom as well as the couple-stress tensor. Similar to [29], in this study, a general power series expansion is assumed for the axial displacement and two other displacements, i.e. the lateral and transverse displacements are approximated according to the classical beam theories. These assumptions may crucially contribute towards revealing the true nature of couple-stress tensor and revising the boundary conditions.

It is noted that in modified couple stress analysis of Timoshenko micro-beams, the slope also appeared as a natural boundary condition [21]. Based on these boundary conditions, [24] derived two distinct closed-form solutions for the micro-cantilever beams, namely the partially clamped solution and the fully clamped solution. They also ignored the contribution of couple stresses to the slope variation and developed an approximate model by discarding a slope from the analysis. However, Dehrouyeh-Semnani and Bahrami [30] showed that such a model cannot provide accurate results when the size of a micro-beam becomes comparable with the material length scale parameter. The second novelty of this study is discarding the slope variation without considering any approximation. As a result, a more convenient form of boundary conditions is developed for Timoshenko micro-beams. Through numerical examples, the results of this model are compared with the results of conventional modified couple stress and classical models and the accuracy of the model is demonstrated.

Here introduces the paper, and put a nomenclature if necessary, with the same font size as the rest of the paper. The paragraphs continue from here and are only separated by headings, subheadings, images and formulae. The section headings are arranged by numbers, bold and 10 pt. Here follows further instructions for authors.

2. The basic arguments in couple stress theory

The beam-like elements have widespread use in modern micro- and nano-electromechanical systems [1]. A series of experiments [2, 3] showed that the mechanical behavior of thin beams in micron and sub-micron scales is strongly size-dependent and cannot be described by the classical elasticity theory. So, the higher-order theories with additional degrees of freedom and intrinsic length-related parameters are developed for the analysis of small-scale structures. Shariati et al. 2021 [4] developed a molecular dynamic analysis for calibration of length-related parameters in the non-classical theories.

The nonlocal elasticity theory [5] has been widely used investigating the size-dependent mechanical behavior of nano-beams [6, 7], nano-disks [8] and nano-plates [9]. So, the strain gradient theory [10] is also used for the analysis of small-scale structures [11-13]. In order to reduce the additional variables and the number of required length-related constants, the original couple stress theory [14-16] developed based on a general non-symmetric couple-stress tensor. In special case of linear isotropic materials, only two extra length-related constants were required in the original couple stress theory to capture the size effect of microstructures.

Yang et al. [17] proposed an artificial equilibrium equation for couple moments and developed the modified couple stress theory (M-CST) based on a symmetric couple-stress tensor and only one required length scale parameter. Therefore, M-CST has been widely used for the analysis of microstructures. (see e.g., [18-21]). However, the equilibrium of couple moments which is used to imply the symmetry of the couple-stress tensor is not deduced from any accepted principle in continuum mechanics [22]. Hadjesfandiari and Dargush [23] proposed an alternative consistent couple stress theory (C-CST) based on a skew-symmetric couple-stress tensor. Recently, C-CST has been widely used for the analysis of isotropic and FGM micro/nano-beams [24-27]. However, Münch, Neff et al. [28] raised serious doubts on validity of the main arguments proposed by [23].

In order to develop a simplified micropolar beam theory, Ramezani et al. [29] assumed power series expansions for both axial displacement and the non-zero micro-rotation. It is noted that in the couple stress theory, the rotation vector is completely dependent on the displacement field and any approximation considered for the displacements, may affect the characteristics of the rotational degree of freedom as well as the couple-stress tensor. Similar to [29], in this study, a general power series expansion is assumed for the axial displacement and two other displacements, i.e., the lateral and transverse displacements are approximated according to the classical beam theories. These assumptions may crucially contribute towards revealing the true nature of couple-stress tensor and revising the boundary conditions.

It is noted that in modified couple stress analysis of Timoshenko micro-beams, the slope also appeared as a natural boundary condition [21]. Based on these boundary conditions, [24] derived two distinct closed-form solutions for the micro-cantilever beams, namely the partially clamped solution and the fully clamped solution. They

also ignored the contribution of couple stresses to the slope variation and developed an approximate model by discarding a slope from the analysis. However, Dehrouyeh-Semnani and Bahrami [30] showed that such a model cannot provide accurate results when the size of a micro-beam becomes comparable with the material length scale parameter. The second novelty of this study is discarding the slope variation without considering any approximation. As a result, a more convenient form of boundary conditions is developed for Timoshenko micro-beams. Through numerical examples, the results of this model are compared with the results of conventional modified couple stress and classical models and the accuracy of the model is demonstrated.

3. The basic arguments in couple stress theory

As the main hypothesis in the couple stress theory, both displacements u_i and rotations θ_i are considered as the degrees of freedom for the material points within a microstructure. The rotation vector in this theory is defined as

$$\theta_i = \frac{1}{2} e_{ijk} u_{k,j} \tag{1}$$

where e_{ijk} is the alternating tensor and the comma denotes the spatial derivation.

From the kinetic point of view, the surface interaction of the material points is resulted from the force traction vector t_i and couple traction vector m_i given by

$$t_i = \sigma_{ji} n_j \tag{2}$$

$$m_i = \mu_{ji} n_j \tag{3}$$

where σ_{ji} and μ_{ji} are general non-symmetric force-stress and couple-stress tensors, respectively, and n_i is the normal unit vector to the bounding surface of material points.

For a continuum of volume V and boundary surface S, the principle of virtual work gives the following equation for couple stress theory [1]

$$\int_V \sigma_{(ji)} \delta\varepsilon_{ij} dV + \int_V \mu_{ji} \delta\theta_{i,j} dV = \int_V F_i \delta u_i dV + \int_S t_i \delta u_i dS + \int_S m_i \delta\theta_i dS \tag{4}$$

where F_i is the body force vector, ε_{ij} is the conventional strain tensor, and $\sigma_{(ji)}$ is the symmetric part of the general force-stress tensor σ_{ji} .

It is noted from Eq.(4) that the boundary conditions in the couple stress theory are expressed in terms of displacements u_i, rotations θ_i and the traction vectors t_i and m_i , respectively. However, Koiter [2] showed that the normal component of the rotation vector cannot be considered an independent natural boundary condition, and the rotation vector should be decomposed to the following normal part $\theta_i^{(n)}$ and the tangential part $\theta_i^{(t)}$

$$\theta_i^{(n)} = \left(\theta_k n_k\right) n_i \tag{5}$$

$$\theta_i^{(t)} = \theta_i - \left(\theta_k n_k\right) n_i \tag{6}$$

As a result, the surface integral corresponding to the rotational degree of freedom can be rewritten as

$$\begin{aligned}\int_S m_i \delta\theta_i dS &= \int_S m_i \delta\theta_i^{(n)} ds + \int_S m_i \delta\theta_i^{(t)} dS \\ &= \int_S \mu_{ji} n_j n_i \left(n_k \delta\theta_k\right) ds + \int_S m_i \delta\theta_i^{(t)} dS\end{aligned} \tag{7}$$

where m_i and $\delta\theta_i^{(n)}$ are replaced, according to Eqs. (3) and (5). Then Eq. (1) and Stoke's theorem are used to show

$$\int_S (\mu_{ji} n_j n_i) n_k \delta\theta_k dS = \int_S (\mu_{ji} n_j n_i) n_k \frac{1}{2} e_{kmn} \delta u_{n,m} dS$$
$$= \oint_C \frac{1}{2} (\mu_{ji} n_j n_i) v_k \delta u_k dc - \int_S \frac{1}{2} e_{kmn} n_k (\mu_{ji} n_j n_i)_{,m} \delta u_n dS \tag{8}$$

where v_n is the unit vector tangent to the curve C that bounds the boundary surface S [3]. This is a very important equation and its contribution to the couple stress analysis of micro-beams will be discussed in the next section.

3. A new approach for couple stress analysis for the micro-beams

As observed in the previous section, only symmetrical part of the force-stress tensor contributes the Eq. (4). However, in original couple stress theory [2, 4, 5] a general non-symmetrical couple-stress tensor μ_{ji} appears in the virtual work formulation. So, in order to develop a feasible theory, some restrictions should be plausibly imposed on μ_{ji} .

In engineering literature, microstructure models are frequently developed based on an approximate form of deformation. Eringen [6] and Ramezani et al. [7] approximated the displacement components by the power series expansion method and derived the first-order micropolar plate and beam models. In this analysis, we followed this approach for couple stress beam theory and approximated the axial component of the displacement field by the following power series expansion

$$u_x(x,z,t) \approx \sum_{j=0}^{N} z^j \phi_{(j)}(x,t) \tag{9}$$

where $\phi_{(j)}$ are undetermined basic functions which depend on the kinematics of deformation, N is the number of terms considered for approximation, and z is the distance from the mid-plane (See Fig. 1). Similar to classical beam theories, the transverse displacement component u_z is simply approximated with only one term, that is

$$u_z(x,z,t) \approx w(x,t) \tag{10}$$

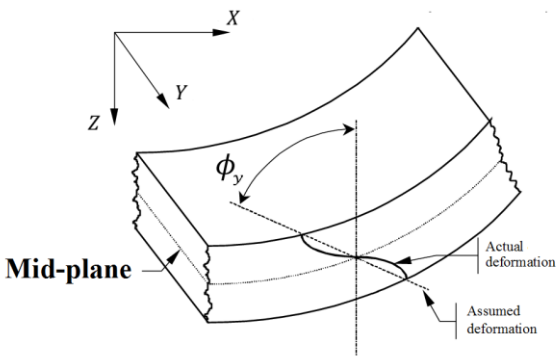

Fig. 1. The cross-section of the deformed structure

Based on the approximated displacements (9) and (10), the non-zero components of the strain tensor ε_{ij} are

$$\varepsilon_{xx} = \sum_{j=0}^{N} z^j \frac{\partial \phi_{(j)}}{\partial x} \tag{11}$$

$$\varepsilon_{xz} = \frac{1}{2}\left\{\sum_{j=0}^{N} j\left(z^{j-1}\right)\phi_{(j)}\right\} + \frac{1}{2}\frac{\partial w}{\partial x} \tag{12}$$

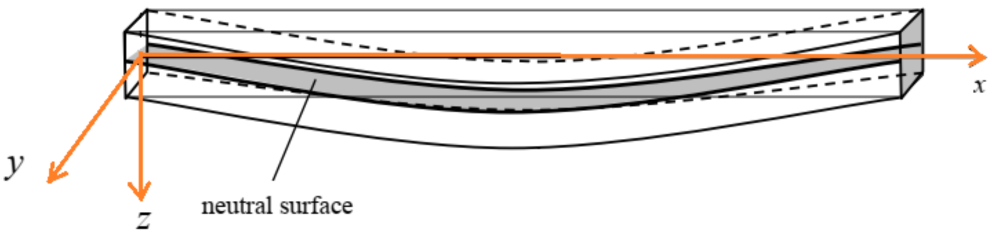

Fig. 2. One-dimensional analysis of beam-like structures

Unlike the micropolar theory, in couple stress theory the rotations θ_i, as another degree of freedom, are dependent on the displacement field u_i (see Eq. (1)). As a result, other than the strain tensor ε_{ij}, the rotation gradient tensor $\theta_{i,j}$ and the couple-stress tensor μ_{ji} are also affected by the approximations (9) and (10). So, considering some specific kinematical assumptions for the deformation, may impose plausible constraints on general non-symmetric couple-stress tensor μ_{ji}.

According to Eqs (1), (9) and (10), components of the rotation vector are

$$\theta_y = \frac{1}{2}\left\{\sum_{j=0}^{N} j\left(z^{j-1}\right)\phi_{(j)}\right\} - \frac{1}{2}\frac{\partial w}{\partial x} \quad , \quad \theta_x = \theta_z = 0 \tag{13}$$

In the one-dimensional analysis of beams, the normal vectors to the cross-sections are assumed only directed along the x–axis (see Fig. 2), i.e., $n_x = \pm 1 \quad n_y = n_z = 0$. So, one can easily show that

$$\theta_k n_k = 0 \tag{14}$$

It is noted from Eq. (5) the normal part of the rotation vector vanishes, and according to Eq. (8) the following equation is obtained for the micro-beams

$$\oint_C \frac{1}{2}\left(\mu_{ji}n_j n_i\right)v_k \delta u_k dc - \int_S \frac{1}{2} e_{kmn} n_k \left(\mu_{ji}n_j n_i\right)_{,m} \delta u_n dS = 0 \tag{15}$$

Thus, the following equation should be satisfied on the boundary sections

$$\mu_{ji} n_j n_i = 0 \tag{16}$$

It is noted also that by adopting displacements in form of (9) and (10), the couple-stress tensor μ_{ji} cannot generate the normal rotation $\theta_i^{(n)}$ on any arbitrary bounding surface inside the domain. As a result, Eq. (16) is also valid inside the volume. Since the second-order tensor $n_i n_j$ is symmetric, it is concluded that the couple-stress tensor "*in analysis of micro-beams*" is skew-symmetric, i.e.,

$$\mu_{ji} = -\mu_{ij} \tag{17}$$

In this approach, without considering any special material behavior, the skew-symmetry of the couple-stress tensor is concluded from Eqs. (9) and (10). Based on a skew-symmetric couple-stress tensor, Hadjesfandiari and Dargush [1] derived the variation in internal energy δU, and the constitutive equations as

$$\delta U = \int_V \left(\sigma_{(ji)}\delta\varepsilon_{ij} + \mu_{ji}\delta\kappa_{ij}\right) dV \tag{18}$$

$$\sigma_{(ji)} = \lambda \varepsilon_{kk}\delta_{ij} + 2G\,\varepsilon_{ij} \tag{19}$$

$$\mu_{ji} = 2Gl^2 \kappa_{ij} \tag{20}$$

where λ and G are the Lame constants, l is the only material length scale parameter, and κ_{ij} is the skew-symmetric curvature tensor, defined as

$$\kappa_{ij} = \frac{1}{2}\left(\theta_{i,j} - \theta_{j,i}\right) \tag{21}$$

It is also noted from Eqs. (12) and (13) that the slope at each section of the beam, can be expressed in the following form

$$\frac{\partial w}{\partial x} = \varepsilon_{xz} - \theta_y \tag{22}$$

It is shown in the next section that Eq. (22) may be used to revise the boundary conditions of couple stress beam models into a more convenient form.

3.1. A couple stress Timoshenko beam model with new boundary conditions

In this section, a couple stress model is developed based only on two first terms of power series expansions (9), corresponding to $j = 0$ and $j = 1$. As a result, the displacement components, which are identical to those in classical Timoshenko beam theory, are noted to be

$$u_x = u\left(x,t\right) - z\,\beta\left(x,t\right) \qquad u_z = w\left(x,t\right) \tag{23a,b}$$

where $\phi_{(0)} = u$, $\phi_{(1)} = -\beta$ and w , are the essential variables of the problem. The non-zero components of the strain tensor and rotation vector are:

$$\varepsilon_{xx} = \frac{\partial u}{\partial x} - z\,\frac{\partial \beta}{\partial x} \quad , \quad \varepsilon_{xz} = \frac{1}{2}\left(-\beta + \frac{\partial w}{\partial x}\right) \quad , \quad \theta_y = -\frac{1}{2}\left(\beta + \frac{\partial w}{\partial x}\right) \tag{24a-c}$$

For the special case of the displacement field (23), the non-zero components of tensor κ_{ij} are found to be

$$\kappa_{xy} = -\kappa_{yx} = \frac{1}{4}\left(\frac{\partial \beta}{\partial x} + \frac{\partial^2 w}{\partial x^2}\right) \tag{25}$$

It is noted from (24) and (25) that Eq. (18) takes the following form for a Timoshenko micro-beam

$$\begin{aligned}
\delta U &= \int_V \left(\sigma_{(xx)}\delta\varepsilon_{xx} + 2\sigma_{(xz)}\delta\varepsilon_{xz} + 2\mu_{xy}\delta\kappa_{yx}\right)dV \\
&= \int_0^L \left[\left(-\frac{\partial N_\sigma}{\partial x}\right)\delta u + \left(-\frac{\partial M_\sigma}{\partial x} - \frac{1}{2}\frac{\partial M_\mu}{\partial x} - Q_\sigma\right)\delta\beta + \left(-\frac{\partial Q_\sigma}{\partial x} + \frac{1}{2}\frac{\partial^2 M_\mu}{\partial x^2}\right)\delta w\right]dx \\
&+ \left[\left(N_\sigma\right)\delta u + \left(M_\sigma + \frac{1}{2}M_\mu\right)\delta\beta + \left(\frac{1}{2}M_\mu\right)\delta\frac{\partial w}{\partial x} + \left(Q_\sigma - \frac{1}{2}\frac{\partial M_\mu}{\partial x}\right)\delta w\right]_{x=0}^{x=L}
\end{aligned} \tag{26}$$

where L is the length of the beam, and the stress resultants across the cross-section A are defined as

$$N_\sigma = \int_A \sigma_{xx} dA \quad , \quad M_\sigma = \int_A -z\sigma_{xx} dA \quad , \quad Q_\sigma = k_s \int_A \sigma_{(xz)} dA \quad , \quad M_\mu = \int_A -\mu_{xy} dA \tag{27a-d}$$

In Eq. (27c), k_s is the shear correction factor corresponding to the improper distribution of shear strains over the beam thickness. The virtual work done by external loads δW takes the following form for the micro-beams [8]

$$\delta W = \int_0^L \left[f\left(x,t\right)\delta u + q\left(x,t\right)\delta w\right]dx + \left[\bar{N}\,\delta u + \bar{M}\,\delta\beta + \bar{Q}\,\delta w\right]_{x=0}^{x=L} \tag{28}$$

where f and q are the components of distributed body force per unit length along the $x-$ and $z-$axis, and $\bar{N}$, $\bar{Q}$, $\bar{M}$ are the external axial force, transverse force and bending moment at the boundary points, respectively.

According to the principle of minimum total potential energy [9]

$$\delta U - \delta W = 0 \tag{29}$$

Substituting (26) and (28) into Eq. (29), gives the following equation

$$\int_0^L \left[\left(-\frac{\partial N_\sigma}{\partial x} - f \right)\delta u + \left(-\frac{\partial M_\sigma}{\partial x} - \frac{1}{2}\frac{\partial M_\mu}{\partial x} - Q_\sigma \right)\delta\beta + \left(-\frac{\partial Q_\sigma}{\partial x} + \frac{1}{2}\frac{\partial^2 M_\mu}{\partial x^2} - q \right)\delta w \right] dx +$$
$$\left[\left(N_\sigma - \overline{N} \right)\delta u + \left(M_\sigma + \frac{M_\mu}{2} - \overline{M} \right)\delta\beta + \left(\frac{M_\mu}{2} \right)\delta\frac{\partial w}{\partial x} + \left(Q_\sigma - \frac{1}{2}\frac{\partial M_\mu}{\partial x} - \overline{Q} \right)\delta w \right]\Bigg|_{x=0}^{x=L} = 0 \tag{30}$$

3.2. Deriving new boundary conditions

In this study, the deformation is described by the basic functions u , w and β , which are the essential variables defined in Eq. (23). However in Eq. (30) the inessential slope variation $\delta\left(\partial w / \partial x\right)$, is also appeared as a natural boundary condition. This implies that Eq. (30) may be revised into a more convenient form. It is noted from Eq. (22) that the slope $\partial w/\partial x$ at each section of the beam can be expressed in terms of shear strain ε_{xz} and bending rotation θ_y . The strain tensor ε_{ij} and the rotation vector θ_i (as the symmetric and skew-symmetric parts of displacement gradient tensor $u_{i,j}$, respectively) are completely independent tensors. So, zero slope variation, (i.e., $\delta\left(\partial w/\partial x\right) = 0$), requires

$$\delta\theta_y = 0 \quad , \quad \delta\varepsilon_{xz} = 0 \tag{31a,b}$$

However, according to Eq. (24), and for $\delta\left(\partial w/\partial x\right) = 0$, the following set of equations is obtained

$$\delta\theta_y = -\frac{1}{2}\left(\delta\beta + \delta\frac{\partial w}{\partial x} \right) = -\frac{1}{2}\left(\delta\beta + 0 \right) = 0 \tag{32}$$

$$\delta\varepsilon_{xz} = \frac{1}{2}\left(-\delta\beta + \delta\frac{\partial w}{\partial x} \right) = \frac{1}{2}\left(-\delta\beta + 0 \right) = 0 \tag{33}$$

As can be seen, both $\delta\theta_y$ and $\delta\varepsilon_{xz}$ vanish, if the constraint $\delta\beta = 0$ is satisfied. This means that in sections with zero slopes, no rotation occurs. As a result, the boundary condition $\delta\left(\partial w / \partial x\right)$ can be replaced by $\delta\beta$ and the rotational boundary conditions that appeared in Eq. (30), may be written in the following form

$$\left[\left(M_\sigma + \frac{M_\mu}{2} - \overline{M} \right)\delta\beta + \left(\frac{M_\mu}{2} \right)\delta\frac{\partial w}{\partial x} \right]\Bigg|_{x=0}^{x=L} = \left[\left(M_\sigma + M_\mu - \overline{M} \right)\delta\beta \right]\Bigg|_{x=0}^{x=L} \tag{34}$$

For the arbitrary amounts of δu , $\delta\beta$ and δw , the equilibrium equations are noted to be

$$-\frac{\partial N_\sigma}{\partial x} - f\left(x,t\right) = 0 \tag{35}$$

$$-\frac{\partial M\left(x,t\right)}{\partial x} - Q\left(x,t\right) = 0 \tag{36}$$

$$-\frac{\partial Q\left(x,t\right)}{\partial x} - q\left(x,t\right) = 0 \tag{37}$$

where Q and M as the total transverse force and bending moment at each arbitrary section of the beam, respectively, are

$$Q = Q_\sigma - \frac{1}{2}\frac{\partial M_\mu}{\partial x} \quad , \quad M = M_\sigma + M_\mu \tag{38a,b}$$

and the boundary conditions are noted to be

$$N_\sigma = \overline{N} \quad or \quad \delta u = 0 \quad at \quad x = 0 \quad and \quad x = L$$

$$M = \overline{M} \quad or \quad \delta\beta = 0 \quad at \quad x = 0 \quad and \quad x = L$$

$$Q = \overline{Q} \quad or \quad \delta w = 0 \quad at \quad x = 0 \quad and \quad x = L \tag{39a-c}$$

4. Numerical investigations

In this section, the new model with revised boundary conditions is used to for the analysis of micro-beams. In some examples, micro-beams with different loadings and support conditions are investigated to show the accuracy and effectiveness of the proposed model. Then, some insufficiencies in M-CST models are discussed and it is shown that the current study provides more convenient tool for size-dependent analysis of micro-beams.

According to the basic arguments considered in this study, total bending moment and transverse shear force at each section are given in Eq. (38). Substituting (19), (20), (24) and (25), into Eq. (38) gives the following equation for the bending moment and the transverse force in terms of generalized displacements β and w

$$M(x) = EI\frac{d\beta}{dx} + \frac{1}{2}GAl^2\left(\frac{d^2w}{dx^2} + \frac{d\beta}{dx}\right) \tag{40}$$

$$Q(x) = k_s GA\left(\frac{dw}{dx} - \beta\right) - \frac{1}{4}GAl^2\left(\frac{d^3w}{dx^3} + \frac{d^2\beta}{dx^2}\right) \tag{41}$$

where I denotes the second-moment area of the cross-section.

4.1. micro-cantilever Timoshenko loaded at free end

For the loading and coordinates system shown in Fig. 3, the transverse force and bending moment at each section of the micro-cantilever beam, are P and $P(L-x)$, respectively. As a result, the following differential equations are obtained for the problem

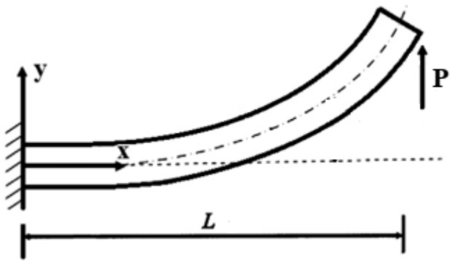

Fig. 3. The micro-cantilever beam clamped at $x = 0$ and subjected to a concentrated load at its free end.

$$\frac{EI}{2}(\Theta' - \gamma') + \frac{GAl^2}{2}\Theta' = P(L-x) \tag{42}$$

$$k_s GA\gamma - \frac{GAl^2}{4}\Theta'' = P \tag{43}$$

where

$$\gamma(x) = \frac{dw}{dx} - \beta \qquad\qquad \Theta(x) = \frac{dw}{dx} + \beta \tag{44a,b}$$

and prime symbol represents the derivation with respect to x .

For the clamped end, the section cannot endure any shear or rigid rotation. So, it can be easily concluded that

$$\Theta(0) = \gamma(0) = 0 \tag{45}$$

At the free end of the beam, the total bending moment is zero. So, it is noted from Eq. (42) that the necessary conditions for satisfaction of $M\left(L\right)=0$, are

$$\Theta'\left(L\right)=\gamma'\left(L\right)=0 \tag{46}$$

Using the boundary conditions (45) and (46), the following closed-form solutions can be found for the set of Eqs. (42) and (43)

$$\gamma\left(x\right)=-\gamma_p\left(\frac{\cosh\Pi\left(L-x\right)}{\cosh\Pi L}-1\right) \tag{47}$$

$$\Theta\left(x\right)=-\frac{EI}{EI+GAl^2}\gamma_p\left(\frac{\cosh\Pi\left(L-x\right)}{\cosh\Pi L}-1\right)+\frac{2P}{EI+GAl^2}\left(Lx-\frac{1}{2}x^2\right) \tag{48}$$

where

$$\Pi=\frac{2}{l}\sqrt{\frac{k_s\left(EI+GAl^2\right)}{EI}} \qquad \gamma_p=\frac{P}{2k_sG}\frac{2EI+GAl^2}{\left(GAl^2\right)\left(EI\right)} \tag{49a,b}$$

The rotation β and the slope dw/dx can then be obtained from Eq. (44). It is noted from Fig. 3 that $w\left(0\right)=0$. So, the rotation and transverse deflection at each section of the beam is found in the following forms

$$\beta\left(x\right)=\frac{1}{2}\frac{GAl^2}{EI+GAl^2}\gamma_p\left(1-\frac{\cosh\Pi\left(L-x\right)}{\cosh\Pi L}\right)+\frac{P}{EI+GAl^2}\left(Lx-\frac{x^2}{2}\right) \tag{50}$$

$$w\left(x\right)=\frac{1}{2}\frac{2EI+GAl^2}{EI+GAl^2}\gamma_p\left(x-\frac{\tanh\Pi L}{\Pi}+\frac{1}{\Pi}\frac{\sinh\Pi\left(L-x\right)}{\cosh\Pi L}\right)+\frac{P}{EI+GAl^2}\left(L\frac{x^2}{2}-\frac{x^3}{6}\right) \tag{51}$$

As an example, a circular micro-beam of diameter d and length $L=5d$ is assumed to be loaded with a concentrated load $P=100\mu N$ at its free end. The mechanical properties of constructing material are $E=1.44GPa$, $\nu=0.38$ and $l=17.6\mu m$ [10]. In this case, the shear correction factor, introduced in Eq. (27c), takes values close to unity, i.e. $k_s=1$ [11].

For different ratios of diameter to material length scale parameter d/l, results of the present model for the maximum deflection at the tip point are illustrated in Fig. 4 and compared with results of the classical Timoshenko theory. It can be seen that for lower diameters, there is a considerable difference between the results of the presented model and the classical solution. However, as the diameter increases, the deviation of couple stress solution from the classical solution diminishes. This indicates that the size effect is significant when the diameter is close to the amount of material length scale parameter l.

Fig. 5 shows the results of the present model and classical solution for the rotations at the tip point. Similar to what was observed for the deflections; the presented couple stress model predicts fewer rotations for the beams with lower diameters. It is also observed that the deviation from classical predictions diminishes as the diameter increases. So the model can properly capture the size effects.

In order to justify the presented model, the deflections obtained from Eq. (51) are compared in Fig. 6 with the results of M-CST Timoshenko model [12] and the results of the M-CST Euler-Bernoulli model [13]. As can be seen, results of the current study are in complete agreement with the results of the first-order shear deformation model. However, a slight difference is observed between the results of current study and those given by the non-shear model. This demonstrates the validity of the presented model. As another example, a rectangular micro-cantilever beam of width b and thickness h is considered. For the loading condition shown in Fig. 3, results of the present model for the bending rigidity are shown in Fig. 7. The bending rigidity is defined as the ratio of maximum defection $w_{\max}$ to the applied load P. Since a variety of dimensionless thicknesses h/l are considered, the cross-sectional aspect ratio and the length of the beam are set to be $b/h=2$ and $L=20h$, respectively, for keeping the shape of the micro-beam. By ignoring the Poisson effect, the shear correction factor is considered to be $k_s=5/6$ [14]. The results of a strain gradient model developed by [10] are also given in Fig. 7 for the sake of comparison. It can be observed that the present model provides sufficiently accurate results compared with a more sophisticated strain gradient model. The difference between the results is significant only when the thickness is near the material length scale parameter, i.e., $h \approx l$. However, the results of both models become convergent for $h > 4l$. This is an important limit and if the thickness of the beam is less than that, the analysis should be done by the classical theory.

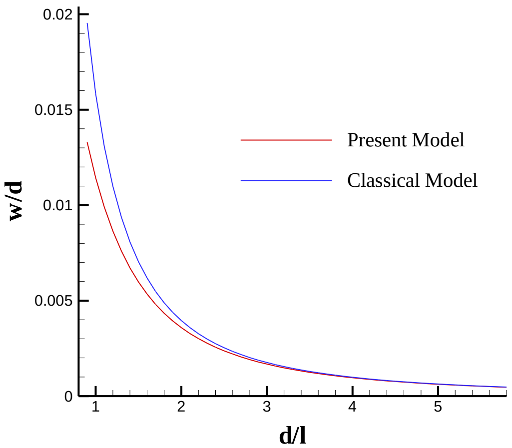

Fig. 4. The results of present model and the classical solution for normalized deflection at tip point.

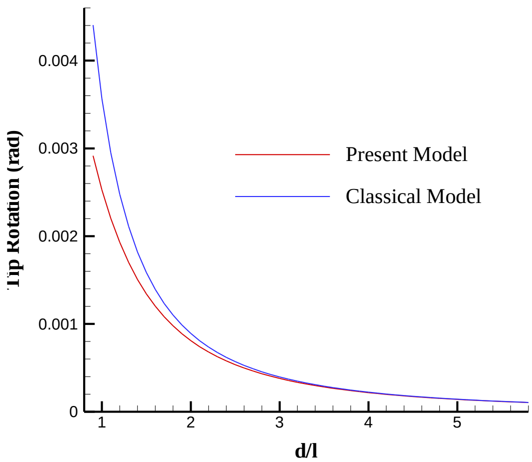

Fig. 5. The results of the present model and the classical solution for the maximum rotation at the tip point.

4.2. Insufficiencies of M-CST boundary conditions at clamped ends

In this section, we show that the boundary conditions in modified couple stress may result in unacceptable solutions for Timoshenko micro-cantilever beams. According to a model developed by [8], the following boundary equation appeared in the variational formulation of the Timoshenko micro-beam (see also Eq. (30))

$$\left[\frac{1}{2} M_{\mu} \delta\left(\frac{\partial w}{\partial x}\right)\right]\Bigg|_{x=0}^{x=L} = 0 \qquad (52)$$

Based on Eq. (52), Alavi et al. [11] derived two distinct solutions for the micro-beam shown in Fig. 3; the "*fully clamped solution*" corresponding to the following boundary condition

$$\frac{\partial w}{\partial x}(0) = 0 \qquad (53)$$

and the "*partially clamped solution*" for

$$M_{\mu}(0) = 0 \qquad (54)$$

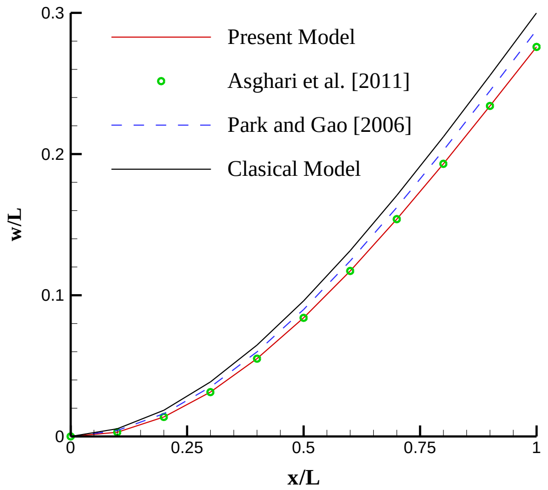

Fig. 6. Comparison of normalized deflection of a micro-beam under a concentrated shear force

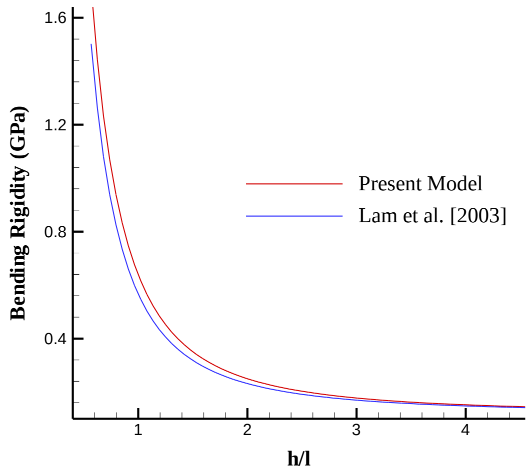

Fig. 7. Bending rigidity of the rectangular micro-beam obtained from the present model and strain gradient theory.

However in this study, a different set of boundary conditions are proposed in Eq. (39). As a result, the present model gives a unique solution for the beams with clamped ends. For a circular micro-beam of length $L = 5d$ and diameter $d = 5l$, the results of fully and partially clamped solutions derived in [11] are illustrated in Fig. 8. Results of the present model and the classical solution are also given for comparison. As can be seen, for a beam of length $L = 5d$ and diameter $d = 5l$, the size effects are negligible and the results of all models are very close to each other. So, both fully and partially clamped solutions give almost the same results in this case.

According to Figs 6 and 7, we consider a case with severe size-dependent effects. For a micro-beam of diameter $d = l$, the results of the fully clamped solution, partially clamped solution and present model are depicted in Fig. 9. It can be observed that the results of the present model are completely identical to the fully clamped solution, but results of the partially clamped solution are found to be significantly different. Now we examine the acceptability of the partially clamped solution. It is also observed that for the points near the fixed end, the partially clamped solution predicts higher deflections even than those in the classical Timoshenko beam model. It seems that there is something wrong with the assumption of partially clamped. For different diameters, the results of the partially clamped solution and the classical solution for deflections are given in Table 1. It is noted that for all considered diameters, the partially clamped solutions give higher values of deflection for the points near the fixed end. However, it is well known that the microstructures endure lower deflections compared with the predictions of the classical solution. This clearly indicates that the partially clamped solution gives erroneous results. As a result, our basic idea for discarding the slope $\delta\left(\partial w/\partial x\right)$ from the natural boundary conditions seems to be reasonable and leads to a solution with more reliable and acceptable results.

4.3. The models with reduced degrees of freedom

In a series of works, authors tried to discard the boundary equation (52) and consider only the deflection w and the rotation β as the natural boundary conditions [11, 15-30]. It is noted that by considering fewer degrees of freedom at each node, the complexity and computational cost will reduce considerably. Alavi et al, [11] made an approximation in Eq. (30) and by ignoring the contribution of M_{μ} to the slope, discarded $\delta\left(\partial w/\partial x\right)$ and derived a two primary variables (2 PV) couple stress model.

For a circular beam of length $L=0.001m$ and length-to-diameter ratio $L=5d$, the results of the 2 PV model developed by [11] and the present model are depicted in Fig. 10. The results of an exact solution developed in [12] are also given for evaluating the accuracy of the models. It is noted that the results of the current study and the 2 PV model are very close to the results of the exact solution. The deflection of a micro-beam with $L=0.0001m$ and the same length-to-diameter ratio is shown in Fig. 11. In this case the results of the 2 PV model have a great deviation, but the present model still preserves its accuracy. For better comparison, the maximum deflections of beams with various lengths are given in Table 2. It is noted that the approximate 2 PV model gives accurate results just for the cases with moderate or low levels of size-dependency. As the size of micro-beam approaches the material length scale parameter, the accuracy of the 2 PV model is badly deteriorated.

Similar to the procedure discussed in section 4.1, a closed-form solution can also be derived for this case. The beam length and cross-section width are selected as $L=20h$ and $b=2l$, respectively. A uniform distributed load $q=\alpha EI/L^3$ is subjected to the micro-beam. According to [31], $\alpha=20/3$ is used for evaluating the bending deformation of beams with various thicknesses. For $h=20l$, the results of the present model are illustrated in Fig. 13 and compared with the results of the 2 PV model, and an exact solution developed by Ma et al. [8]. As can be seen, all models give the same results and discarding the slope from the formulation does not affect the accuracy. Fig. 14 compares the results for a micro-beam with thickness $h=5l$. In this case, the 2 PV model cannot provide accurate results. For various thicknesses, the maximum deflections of the beam predicted by the models are given in Table 3. It can be easily understood that the approximation proposed by [11] gives valid results only for a limited range of microstructure sizes. However, the present model is still accurate and gives the same results as the exact solution. So, by using the proposed consideration, the number of degrees of freedom at each section of micro-beams reduces but the accuracy is not affected at all.

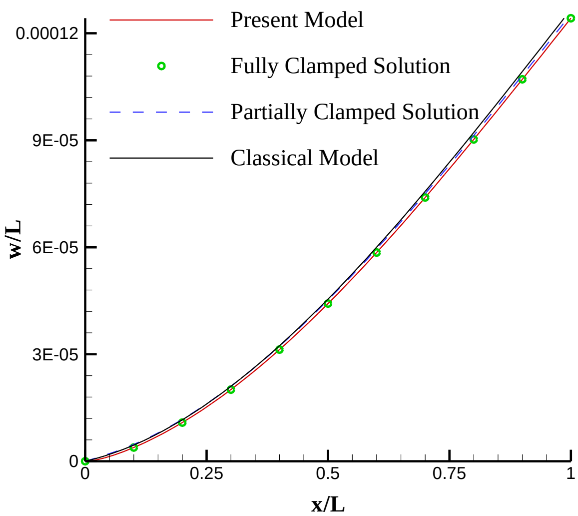

Fig. 8. Deflection of micro-beam with $d/l=5$, obtained from different solutions.

Table 1. The results of different solutions for the deflection w/L of points near the fixed end.

	$\frac{x}{L}$	Present model	Partially clamped solution	Classical model
$\frac{d}{l}=1$	0.1	0.00004957	0.00017240	0.00011323
	0.2	0.00017035	0.00034885	0.00029069
	0.3	0.00034050	0.00029069	0.00052524
$\frac{d}{l}=2$	0.1	0.000018404	0.000034352	0.000028309
	0.2	0.000058177	0.000077409	0.000072674
	0.3	0.000111860	0.000132810	0.000131310
$\frac{d}{l}=3$	0.1	0.000009477	0.000013947	0.000012582
	0.2	0.000028312	0.000033246	0.000032299
	0.3	0.000053361	0.000058571	0.000058360
$\frac{d}{l}=4$	0.1	0.000005752	0.000007529	0.0000070772
	0.2	0.000016578	0.000018463	0.0000181680
	0.3	0.000030912	0.000032876	0.0000328280

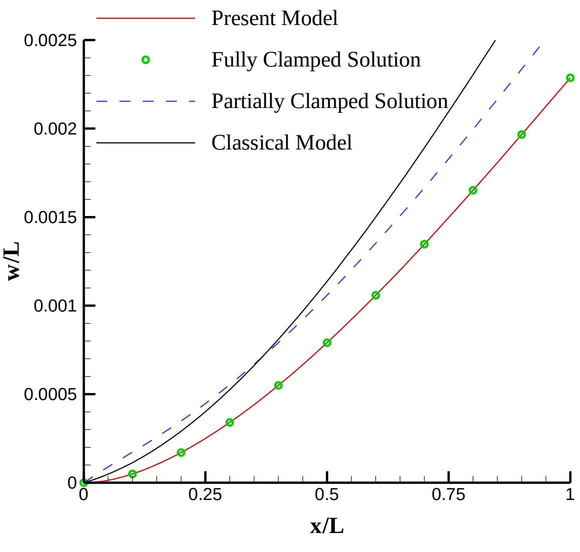

Fig. 9. Deflection of micro-beam with $d/l=1$, obtained from different solutions.

In the case of rectangular simply supported micro-beam shown in Fig. 12, the boundary conditions at the end sections are

$$w\left(0\right)=w\left(L\right)=0 \qquad M\left(0\right)=M\left(L\right)=0 \tag{55a,b}$$

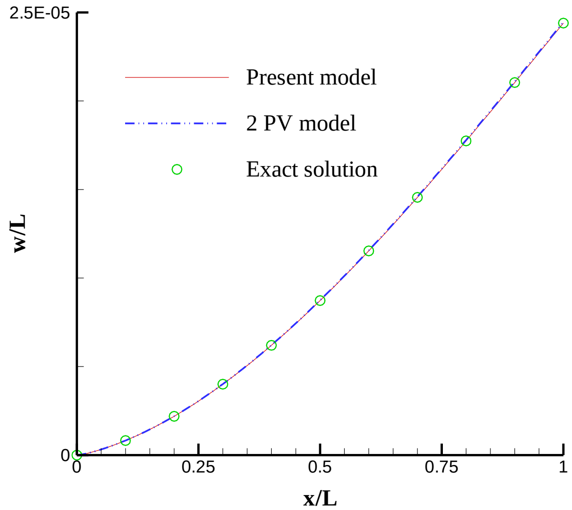

Fig. 10. The results of models with revised boundary conditions for a beam of length $L = 0.001m$.

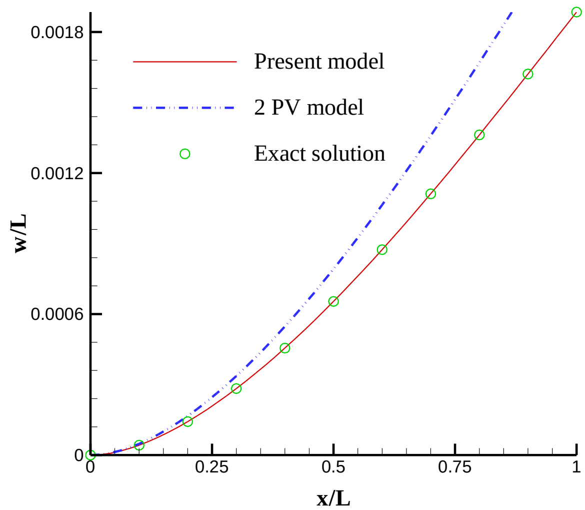

Fig. 11. The results of models with revised boundary conditions for a beam of length $L = 0.0001m$.

Table 2. the maximum deflection $w_{\max}/L$ at the tip point for various beam lengths

	Present model	2 PV model	Exact solution
$L = 5l$	2.0131×10^{-7}	2.6042×10^{-7}	2.0131×10^{-7}
$L = 10l$	1.2629×10^{-7}	1.3611×10^{-7}	1.2629×10^{-7}
$L = 100l$	1.3903×10^{-7}	1.3914×10^{-7}	1.3903×10^{-7}

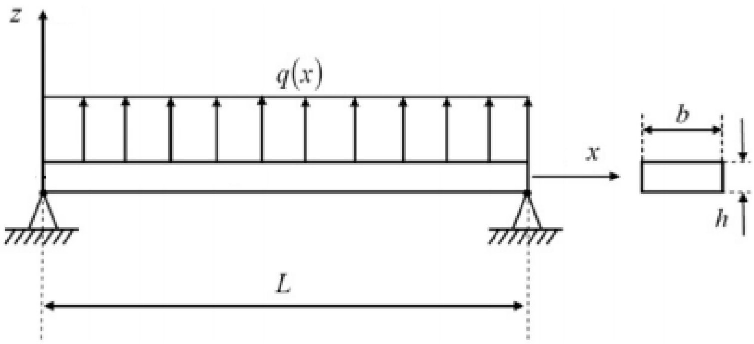

Fig. 12. The simply supported micro-beam subjected to a distributed load $q(x)$.

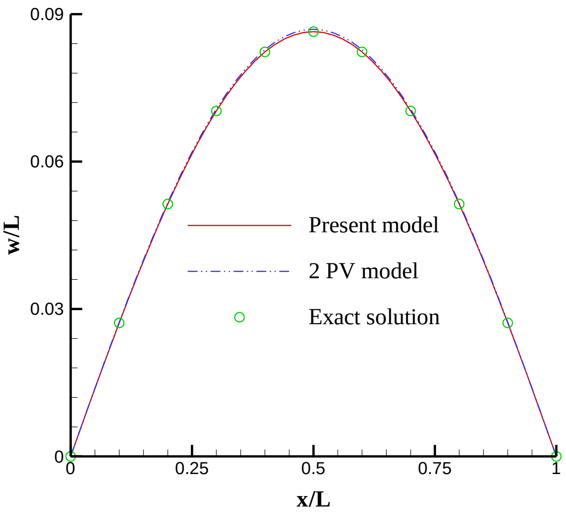

Fig. 13. The results of models with revised boundary conditions for a beam of length $L = 20h$ and thickness $h = 20l$.

4. Summary

In current study a new idea is proposed for the couple stress analysis of micro-beams. The displacement components are approximated by power series expansions of displacements components according to the kinematical assumptions of beam theories. As a result, a decisive conclusion is made and the couple-stress tensor is found to be skew-symmetry. Furthermore, the proposed approach makes it possible to revise the boundary conditions of couple stress beam theories into a more convenient form. In the special case of couple stress Timoshenko beam models, other than the classical δw and $\delta\beta$, the inessential slope variation $\delta(\partial w/\partial x)$, also appears as a natural boundary condition in conventional modified couple stress (M-CST) models. In this study, the slope variation is discarded and a more convenient set of boundary conditions similar to those in classical model is derived without considering any approximation. Several numerical examples are considered and it is shown that the presented model gives exactly the same results compared with available solutions in the literature.

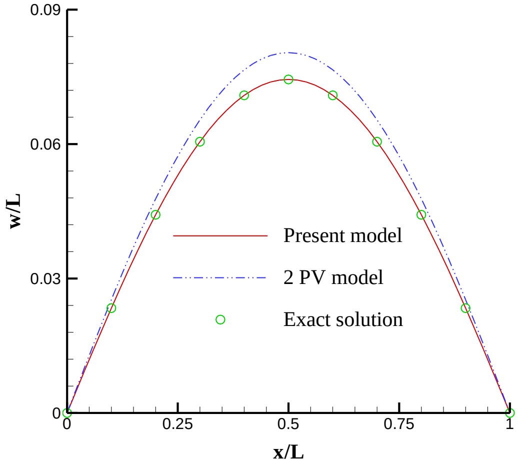

Fig. 14. The results of models with revised boundary conditions for a beam of length $L = 20h$ and thickness $h = 5l$.

Table 3. The maximum deflection $w_{\max}/L$ of simply supported beams for different thicknesses.

	Present model	2 PV model	Exact solution
$h = l$	0.01643	0.02772	0.01643
$h = 5l$	0.07443	0.08041	0.07443
$h = 10l$	0.08374	0.08553	0.08374
$h = 20l$	0.08644	0.08691	0.08644

References

[1] A. Hadjesfandiari, G. Dargush, Couple stress theory for solids, *International Journal of Solids and Structures*, Vol. 48, No. 18, pp. 2496-2510, 2011.

[2] W. Koiter, Couple stresses in the theory of elasticity I,II, *Proceedings of the Koninklijke Nederlandse Akademie Van Wetenschappen, Series BB* Vol. 67, pp. 17-44, 1964.

[3] S. Park, X. Gao, Variational formulation of a modified couple stress theory and its application to a simple shear problem, *Zeitschrift für angewandte Mathematik und Physik*, Vol. 59, No. 5, pp. 904-917, 2008.

[4] R. Mindlin, H. Tiersten, Effects of couple stresses in linear elasticity, *Archive for Rational Mechanics and Analysis*, Vol. 11, pp. 415-447, 1962.

[5] R. Toupin, Elastic materials with couple-stresses, *Archive for Rational Mechanics and Analysis*, Vol. 11, pp. 385-414, 1962.

[6] A. Eringen, Theory of micropolar plates, *Zeitschrift für angewandte Mathematik und Physik ZAMP*, Vol. 18, No. 1, pp. 12-30, 1967.

[7] S. Ramezani, R. Naghdabadi, S. Sohrabpour, Analysis of micropolar elastic beams, *European Journal of Mechanics-A/Solids*, Vol. 28, No. 2, pp. 202-208, 2009.

[8] H. Ma, X. Gao, J. Reddy, A microstructure-dependent Timoshenko beam model based on a modified couple stress theory, *Journal of the Mechanics and Physics of Solids*, Vol. 56, No. 12, pp. 3379-3391, 2008.

[9] D. Steigmann, Equilibrium of prestressed networks, *IMA journal of applied mathematics*, Vol. 48, No. 2, pp. 195-215, 1992.

[10] D. Lam, F. Yang, A. Chong, J. Wang, P. Tong, Experiments and theory in strain gradient elasticity, *Journal of the Mechanics and Physics of Solids*, Vol. 51, No. 8, pp. 1477-1508, 2003.

[11] E. Alavi, M. Sadighi, M. Pazhooh, J. Ganghoffer, Development of size-dependent consistent couple stress theory of Timoshenko beams, *Applied Mathematical Modelling*, Vol. 79, pp. 685-712, 2020.

[12] M. Asghari, M. Kahrobaiyan, M. Rahaeifard, M. Ahmadian, Investigation of the size effects in Timoshenko beams based on the couple stress theory, *Archive of Applied Mechanics,* Vol. 81, No. 7, pp. 863-874, 2011.

[13] S. Park, X. Gao, Bernoulli–Euler beam model based on a modified couple stress theory, *Journal of Micromechanics and Microengineering,* Vol. 16, No. 11, pp. 2355, 2006.

[14] T. Kaneko, On Timoshenko's correction for shear in vibrating beams, *Journal of Physics D: Applied Physics,* Vol. 8, No. 16, pp. 1927, 1975.

[15] M. Kahrobaiyan, M. Asghari, M. Ahmadian, A Timoshenko beam element based on the modified couple stress theory, *International Journal of Mechanical Sciences,* Vol. 79, pp. 75-83, 2014.

[16] M. Mohammadi, A. Farajpour, A. Rastgoo, Coriolis effects on the thermo-mechanical vibration analysis of the rotating multilayer piezoelectric nanobeam, *Acta Mechanica,* Vol. 234, No. 2, pp. 751-774, 2023/02/01, 2023.

[17] M. Mohammadi, A. Farajpour, A. Moradi, M. Hosseini, Vibration analysis of the rotating multilayer piezoelectric Timoshenko nanobeam, *Engineering Analysis with Boundary Elements,* Vol. 145, pp. 117-131, 2022.

[18] M. Mohammadi, A. Rastgoo, Primary and secondary resonance analysis of FG/lipid nanoplate with considering porosity distribution based on a nonlinear elastic medium, *Mechanics of Advanced Materials and Structures,* Vol. 27, No. 20, pp. 1709-1730, 2020.

[19] M. Mohammadi, M. Hosseini, M. Shishesaz, A. Hadi, A. Rastgoo, Primary and secondary resonance analysis of porous functionally graded nanobeam resting on a nonlinear foundation subjected to mechanical and electrical loads, *European Journal of Mechanics-A/Solids,* Vol. 77, pp. 103793, 2019.

[20] M. Mohammadi, A. Rastgoo, Nonlinear vibration analysis of the viscoelastic composite nanoplate with three directionally imperfect porous FG core, *Structural Engineering and Mechanics, An Int'l Journal,* Vol. 69, No. 2, pp. 131-143, 2019.

[21] A. Farajpour, A. Rastgoo, M. Mohammadi, Vibration, buckling and smart control of microtubules using piezoelectric nanoshells under electric voltage in thermal environment, *Physica B: Condensed Matter,* Vol. 509, pp. 100-114, 2017.

[22] A. Farajpour, M. H. Yazdi, A. Rastgoo, M. Loghmani, M. Mohammadi, Nonlocal nonlinear plate model for large amplitude vibration of magneto-electro-elastic nanoplates, *Composite Structures,* Vol. 140, pp. 323-336, 2016.

[23] A. Farajpour, M. H. Yazdi, A. Rastgoo, M. Mohammadi, A higher-order nonlocal strain gradient plate model for buckling of orthotropic nanoplates in thermal environment, *Acta Mechanica,* Vol. 227, pp. 1849-1867, 2016.

[24] M. Mohammadi, M. Safarabadi, A. Rastgoo, A. Farajpour, Hygro-mechanical vibration analysis of a rotating viscoelastic nanobeam embedded in a visco-Pasternak elastic medium and in a nonlinear thermal environment, *Acta Mechanica,* Vol. 227, pp. 2207-2232, 2016.

[25] M. Goodarzi, M. Mohammadi, M. Khooran, F. Saadi, Thermo-mechanical vibration analysis of FG circular and annular nanoplate based on the visco-pasternak foundation, *Journal of Solid Mechanics,* Vol. 8, No. 4, pp. 788-805, 2016.

[26] M. Safarabadi, M. Mohammadi, A. Farajpour, M. Goodarzi, Effect of surface energy on the vibration analysis of rotating nanobeam, 2015.

[27] M. Mohammadi, A. Farajpour, A. Moradi, M. Ghayour, Shear buckling of orthotropic rectangular graphene sheet embedded in an elastic medium in thermal environment, *Composites Part B: Engineering,* Vol. 56, pp. 629-637, 2014.

[28] S. Asemi, A. Farajpour, M. Mohammadi, Nonlinear vibration analysis of piezoelectric nanoelectromechanical resonators based on nonlocal elasticity theory, *Composite Structures,* Vol. 116, pp. 703-712, 2014.

[29] M. Mohammadi, M. Ghayour, A. Farajpour, Free transverse vibration analysis of circular and annular graphene sheets with various boundary conditions using the nonlocal continuum plate model, *Composites Part B: Engineering,* Vol. 45, No. 1, pp. 32-42, 2013.

[30] M. Mohammadi, M. Goodarzi, M. Ghayour, A. Farajpour, Influence of in-plane pre-load on the vibration frequency of circular graphene sheet via nonlocal continuum theory, *Composites Part B: Engineering,* Vol. 51, pp. 121-129, 2013.

[31] W. Xia, L. Wang, L. Yin, Nonlinear non-classical microscale beams: static bending, postbuckling and free vibration, *International Journal of Engineering Science,* Vol. 48, No. 12, pp. 2044-2053, 2010.