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Weak Interaction p-Wave Superfluid Fermi Gas: Low-Temperature Shear Viscosity

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Abstract

The shear viscosity tensor of a superfluid Fermi gas in a *p*-wave state with weak interactions was investigated using the Boltzmann equation at low temperatures. Therefore, transition probabilities of the binary collision, decay, and coalescence processes were considered for this purpose. Theoretically, the dominant processes at low temperatures find to be the binary ones, whereas the elements of the shear viscosity (*i.e.*, $\eta_{xy}, \eta_{xx}, \eta_{yy}$) report to be proportional to $(1/T)^2$. Moreover, η_{xz}, η_{yz} behaved as $(1/T)^4$, and η_{zz} was proportional to $(1/T)^6$.

Keywords: Superfluid, Fermi gas; Shear viscosity.

Introduction

The molecular Bose–Einstein condensate can emerge from weakly bound molecules consisting of two fermion atoms. If the scattering length is positive, the molecular condensations are then formed with respect to a repulsive effective interaction. The atoms pair up when the interaction is attractive, similarly to the way electrons in superconducting metals or the atoms of liquid helium three form Cooper pairs.

There is a transition to the superfluid state in ultracold atomic fermion gases, *e.g.*, ^{6}Li and ^{40}K (1-3). It has been demonstrated that sweeping a Fermi gas of ^{6}Li through a *p* resonance can produce *p*-wave molecules (4). As a result, the ultracold atoms, such as ^{6}Li, can create the *p*-wave pairing super fluids. Ho and Diener created a solution of the gap equation analytically based on a generalization of Nozieres and Schmitt-Rink's potential near the scattering resonance (5, 6). According to their results, the ground state of $l = 1$ superfluid was an orbital ferromagnetic material, and the pairing function had the angular dependence of Y_{11}.

There have been remarkable developments in both theoretical and experimental studies of BCS–BEC crossover in the ultracold trapped Fermi gases. By tuning a variable bias magnetic field near the Feshbach resonance, in an optically trapped Fermi gas of ^{6}Li or ^{40}K, the atomic interaction strength can be controlled (7). Generally, a high-temperature superfluid believed to exist at unitarity of the Fermi temperature order $T_c = 0.167(13)T_F$ (8).

Ultracold Fermi gases can greatly help analyze quantum liquids, for they provide suitable laboratory conditions for modeling and searching processes of testing the quantum theories. The transition to superfluidity can occur at sufficiently low temperatures, and this new phase of matter exhibits very interesting properties. For example, it flows without friction.

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An intriguing characteristic of the ultracold trapped Fermi gases in the BCS regime is their shear viscosity. Taking a temperature gradient or velocity gradient in a slightly inhomogeneous gas can make an energy or momentum fluxes, respectively. Hence, these coefficients of proportionality are thermal conductivity κ and viscosity η, respectively. In the field of calculating the transport properties of ultracold quantum gases, a lot of research have been done. You can read the progress of the last two decades in brief (9).

A cigar-shaped cloud can be observed to expand both below and above the transition temperature to determine the viscous relaxation rate (10-11). Massignan *et al.* calculated the viscous relaxation rate of a uniform Fermi gas in the unitarity limit (12). For this purpose, they used the Boltzmann–Vlasov equation and reported that the viscous relation rate τ_η was proportional to T^{-2} at low temperatures and to $T^{1/2}$ at high temperatures. Within the high temperature range and for the special value of $k_F |a| = 4.5$, the temperature dependency of viscosity was proportional to $T^{-3/2}$.

To understand how a medium affects the scattering matrix of a homogeneous gas in the unitarity limit, Brunn and Smith calculated the viscous and thermal relaxation rates. In comparison with their values without medium, the viscous and thermal relaxation rates increased by almost an order of magnitude (10, 13).

Shahzamanian and Yavari calculated the shear and bulk viscosities of the *s*-wave superfluid ^{6}Li in the weak-coupling limit, for the temperatures approximately T_c and zero (14, 15). In their study, the shear viscosity decreased by $(1 - T/T_c)$ with temperature, while passing through the transition temperature, and then became constant at extremely low temperatures.

Dusling and Schäfer determined the bulk viscosity of a dilute Fermi gas near unitarity by using an effective Lagrangian for the spin-$1/2$ fermions and leading correction that came from a boson loop (16). According to their results, the bulk viscosity increased by $1/(k_F a)$ or T_F/T and depended on temperature as $T^{-5/2}$.

Rupak and Schäfer derived the viscosity of the unitarity ultracold gases. They adopted the Boltzmann equation and indicated that a bosonic superfluid's shear viscosity scales as ξ^5/T^5 at low temperatures in which ξ is the universal parameter relating chemical potential and the Fermi energy ($\mu = \xi\varepsilon_F$) (17).

In another study, a Fermi gas was analyzed experimentally in the unitarity limit, and the temperature dependence of the shear viscosity was obtained both at low and high temperatures. According to the results, at low temperatures, the shear viscosity was constant and dependent on density *i.e.,* $\eta \propto \hbar n$. However, at high temperatures, it was proportional to $T^{3/2}$ (18). It can be seen that other experimental research also confirms these results (19). Moreover, the shear viscosity of an ultracold Fermi gas was estimated as a function of the reduced temperature near the transition temperature T_c. The results indicated that the local shear viscosity decreased rapidly above T_c (20).

Enss *et al.* employed the Kubo formula and calculated the shear viscosity of a fermionic system diagrammatically above the transition temperature. They achieved the temperature and frequency dependency of the shear viscosity (21).

Taylor and Randeria extracted sum rules for the shear and bulk viscosity spectral function of any Bose or Fermi system (22). The shear viscosity of an ultracold Fermi gas were then determined in a BCS–BEC crossover scheme (23). Using the BCS-Legget ground state, Levin's group demonstrated that the shear viscosity reaches zero at $T \to 0$ (24).

In this study, we considered the shear viscosity of a *p*-wave superfluid Fermi gas in the state Y_{11} at low temperatures. For this purpose, we employed a potential model. This model generalizes the separable potential model introduced by Nozieres and Schmitt-Rink for the *s*-wave fermion gases (6). Ho and Diener generalized the model to the *p*-, *d*- and *f*-wave fermion superfluid gases (5). This study focused only on the superfluid's shear viscosity elements. As mentioned earlier, the shear viscosity of the bosonic part was calculated by Rupak and Schäfer (17). We supposed that there were negligible interactions between the superfluid components.

Transition probabilities were obtained from the potential model proposed by Nozieres and Schmitt-Rink. More clearly, this potential model does not depend on the angles between the momenta of quasiparticles. With the help of Sykes and Brooker's procedure, we calculated the shear viscosity elements of superfluid Fermi gas at low temperatures (25).

Materials and Methods

In order to determine the transition probabilities of possible processes at low temperatures, the Boltzmann equation was employed with respect to the effective interactions between quasiparticles. In a superfluid, the number of quasiparticles does not conserve, which allows other processes can occur such as decay processes in which one quasiparticle decays into three as well as coalescence processes in which three quasiparticles coalesce to produce one. However, in a normal Fermi gas at low temperatures, only the binary processes are dominant.

The Bogoliuobov transformations on the normal interaction are used to determine the effective interactions between quasiparticles in the superfluid state.

The potential energy in the BCS Hamiltonian is as follows (26):

$$V = \sum_{\vec{p}_1,\vec{p}_2,\vec{p}_3,\vec{p}_4} V_{\vec{q}}\, a^{\dagger}_{\vec{p}_3,\sigma} a^{\dagger}_{\vec{p}_4,\sigma'} a_{\vec{p}_2,\sigma'} a_{\vec{p}_1,\sigma}, \qquad (1)$$

where $\vec{q} = \vec{p}_3 - \vec{p}_1 = \vec{p}_2 - \vec{p}_4$. The Bogoliubov transformation is defined as the following equation:

$$a_{\vec{p},\sigma} = u_{\vec{p}}\alpha_{\vec{p},\sigma} + \sigma v_{\vec{p}}\alpha^{\dagger}_{-\vec{p},-\sigma}, \qquad (2)$$

where $a^{\dagger}_{\vec{p},\sigma}$ and $a_{\vec{p},\sigma}$ are the normal quasiparticle creation and annihilation operators as well as $\alpha^{\dagger}_{\vec{p},\sigma}$ and $\alpha_{\vec{p},\sigma}$ are the creation and annihilation operators in the superfluid.

The *p*-wave superfluids, have the following properties between $u_{\vec{p}}$ and $v_{\vec{p}}$ (27)

$$u_{-\vec{p}} = u_{\vec{p}} \quad ; \quad v_{-\vec{p}} = -v_{\vec{p}} \ . \qquad (3)$$

Eqs. (2) and (3) are applied in Eq. (1), and the anticommutation relations are used for the quasiparticle creation and annihilation operators. From the following equation, we can find the potential energy (28)

$$V = \sum_{\vec{p}_1,\vec{p}_2,\vec{p}_3,\vec{p}_4} V_{\vec{q}}\Big[(u_{\vec{p}_3}\alpha^{\dagger}_{\vec{p}_3,\sigma} + \sigma v_{\vec{p}_3}\alpha_{-\vec{p}_3,-\sigma})(u_{\vec{p}_4}\alpha^{\dagger}_{\vec{p}_4,\sigma'} + \sigma' v_{\vec{p}_4}\alpha_{-\vec{p}_4,-\sigma'}) \times (u_{\vec{p}_2}\alpha_{\vec{p}_2,\sigma'} + \sigma' v_{\vec{p}_2}\alpha^{\dagger}_{-\vec{p}_2,-\sigma'})(u_{\vec{p}_1}\alpha_{\vec{p}_1,\sigma} + \sigma v_{\vec{p}_1}\alpha^{\dagger}_{-\vec{p}_1,-\sigma})\Big], \qquad (4)$$

it contains $\alpha^{\dagger}_{\vec{p}_3,\sigma}\alpha^{\dagger}_{\vec{p}_4,-\sigma'}\alpha^{\dagger}_{-\vec{p}_2,\sigma'}\alpha_{\vec{p}_1,\sigma}$, $\alpha^{\dagger}_{\vec{p}_3,\sigma}\alpha_{\vec{p}_1,\sigma}\alpha_{-\vec{p}_4,-\sigma'}\alpha_{\vec{p}_2,\sigma'}$, $\alpha^{\dagger}_{\vec{p}_3,\sigma}\alpha^{\dagger}_{\vec{p}_4,\sigma'}\alpha_{\vec{p}_2,\sigma'}\alpha_{\vec{p}_1,-\sigma'}$, $\alpha^{\dagger}_{\vec{p}_3,\sigma}\alpha^{\dagger}_{\vec{p}_4,\sigma'}\alpha^{\dagger}_{-\vec{p}_2,-\sigma'}\alpha^{\dagger}_{-\vec{p}_1,-\sigma}$, and $\alpha_{-\bar{p}_3,-\sigma}\alpha_{-\bar{p}_4,-\sigma'}\alpha_{\bar{p}_2,\sigma'}\alpha_{\bar{p}_1,\sigma}$. Using these terms, a quasiparticle can be converted into three, three quasiparticles can be coalesced into one, two quasiparticles can be converted into two, four quasiparticles can be created from the condensate, and four quasiparticles can be scattered into the condensate, respectively. The last two processes are impossible, for they break the total energy conservation.

In the decay process, transition probability is given as follows:

$$W_{13} = \left|\langle \cdots; \vec{p}_3,\sigma; \vec{p}_4,\sigma'; -\vec{p}_2,-\sigma'; \cdots |V| \cdots; \vec{p}_1,\sigma; \cdots\rangle\right|^2, \qquad (5)$$

hence, the decay transition probability can be derived by substituting Eq. (4) in Eq. (5):

$$W_{13}(\vec{p}_1,\vec{p}_2,\vec{p}_3,\vec{p}_4;\sigma,\sigma') = \frac{1}{4}\Big|(u_{\vec{p}_4}u_{\vec{p}_3}u_{\vec{p}_1}v_{\vec{p}_2} + v_{\vec{p}_4}v_{\vec{p}_3}v_{\vec{p}_1}u_{\vec{p}_2})\big[\sigma'(V_{\vec{p}_3-\vec{p}_1} + V_{\vec{p}_1-\vec{p}_3}) - \sigma'\delta_{\sigma,\sigma'}(V_{\vec{p}_4-\vec{p}_1} + V_{\vec{p}_1-\vec{p}_4})\big]$$
$$+ (u_{\vec{p}_3}u_{\vec{p}_2}u_{\vec{p}_1}v_{\vec{p}_4} + v_{\vec{p}_3}v_{\vec{p}_2}v_{\vec{p}_1}u_{\vec{p}_4})\big[-\sigma'(V_{\vec{p}_3-\vec{p}_1} + V_{\vec{p}_1-\vec{p}_3}) + \sigma'\delta_{\sigma,-\sigma'}(V_{-\vec{p}_1-\vec{p}_2} + V_{\vec{p}_1+\vec{p}_2})\big]$$
$$+ (u_{\vec{p}_4}u_{\vec{p}_2}u_{\vec{p}_1}v_{\vec{p}_3} + v_{\vec{p}_4}v_{\vec{p}_2}v_{\vec{p}_1}u_{\vec{p}_3})\big[\sigma'\delta_{\sigma,\sigma'}(V_{\vec{p}_4-\vec{p}_1} + V_{\vec{p}_1-\vec{p}_4}) - \sigma'\delta_{\sigma,-\sigma'}(V_{-\vec{p}_1-\vec{p}_2} + V_{\vec{p}_1+\vec{p}_2})\big]\Big|^2. \qquad (6)$$

Regarding Eq. (6), it is evident that W_{13} is a function of all momenta through potentials, spins, and temperature. There are other processes whose transition probabilities can be expressed in terms of u and v, such as $W_{31}(\sigma,\sigma')$ and $W_{22}(\sigma,\sigma')$.

The Bogoliubov coefficients $u_{\vec{p}}$ and $v_{\vec{p}}$ are defined as below:

$$u^2_{\vec{p}} = \frac{1}{2}\left(1 + \frac{\varepsilon_{\vec{p}}}{E_{\vec{p}}}\right) \quad ; \quad v^2_{\vec{p}} = \frac{1}{2}\left(1 - \frac{\varepsilon_{\vec{p}}}{E_{\vec{p}}}\right), \qquad (7)$$

where $E^2_{\vec{p}} = \varepsilon^2_{\vec{p}} + \Delta^2_{\vec{p}}$, and $\varepsilon_{\vec{p}}$ denotes the energy of a quasiparticle in the normal state when measured with respect to the chemical potential. Moreover, $\Delta_{\vec{p}}$ indicates the magnitude of the gap in direction $\vec{p}$ on the Fermi surface (27). Furthermore, $\Delta_{\vec{p}} = \Delta(T)\sin\theta_p$ in which $\Delta(T)$ indicates the maximum gap, and θ_p represents the angle between the quasiparticle momentum and the gap axis $\hat{l}$, which is supposed to be along the z-axis.

Since more quasiparticles are gathered in the gap nodes at low temperatures, we have $\sin\theta_{pi} \approx 0$, $\Delta_p \approx 0$, and $E_{\vec{p}} \approx \varepsilon_{\vec{p}} \approx T$. In this temperature region, the Bogoliubov coefficients can be approximated as $u_{\vec{p}} \approx 1$ and $v_{\vec{p}} \approx 0$. Through these approximations, the transition probabilities are defined as below:

$$W_{13}(\uparrow\downarrow) \approx 0 \quad , \quad W_{13}(\uparrow\uparrow) \approx 0,$$
$$W_{31}(\uparrow\uparrow) \approx 0 \quad , \quad W_{31}(\uparrow\downarrow) \approx 0, \qquad (8)$$

$$W_{22}(\uparrow\downarrow) = \frac{1}{4}\left|(V_{\vec{p}_1-\vec{p}_3} + V_{\vec{p}_3-\vec{p}_1})\right|^2, \qquad (9)$$

$$W_{22}(\uparrow\uparrow) = \frac{1}{4}\left|(V_{\vec{p}_1-\vec{p}_3} + V_{\vec{p}_3-\vec{p}_1}) - (V_{\vec{p}_3-\vec{p}_2} + V_{\vec{p}_2-\vec{p}_3})\right|^2, \qquad (10)$$

according to the above equations, the transition probabilities are functions of the potential. In this study, we used the model of potential that was the generalized version of Nozieres and Schmitt-Rink's separable potential. It can be expressed as below (5)

$$V_l(p,p') = \lambda_l W_l(p) W_l(p') \quad , \quad W_l(p) = \frac{\left(\frac{p}{p_0}\right)^l}{\left[1 + \left(\frac{p}{p_0}\right)^2\right]^{\frac{l+1}{2}}}, \qquad (11)$$

where p_0^{-1} denotes the range of potential (we assigned $\hbar \equiv k_B \equiv 1$). Nozieres and Schmitt-Rink considered the Bose condensation of a fermionic gas with an attractive interaction from a weak limit to a strong limit. According to their results, the weak interaction in the *s*-wave fermionic gases can be described by the following potential:

$$V_{p,p'} = \frac{V}{\{(1+\frac{p^2}{p_0^2})(1+\frac{p'^2}{p_0^2})\}^{\frac{1}{2}}}, \qquad (12)$$

where V and p_0^{-1} refer to the strength and range of the potential, respectively. Since the interaction in the *p*-wave Fermi gas analyzed in this study were weak, the potential in (12) was employed to describe the interaction. For the *s*-wave case, eq. (11) was reduced to eq. (12). It is possible to write this potential for the *p*-wave state as the following simplified form:

$$V_1(p_1, p_3) = \lambda_1 \frac{p_0^2 p_1 p_3}{(p_0^2 + p_1^2)(p_0^2 + p_3^2)}. \qquad (13)$$

At low temperatures, quasiparticles gather near the Fermi surface; hence, their wave numbers are close to the Fermi wave number. As a result, the following equation can be considered:

$$V_1(p_1, p_3) = V_1(p_2, p_3) \approx \lambda_1 \frac{p_0^2 p_F^2}{(p_0^2 + p_F^2)^2}. \qquad (14)$$

For estimating this potential, it is essential to know the order of magnitude of p_F and p_0 with respect to each other. Moreover, the kinetic energy of the Bogoliuobov quasiparticles was calculated $p_0^2 / 2m - \varepsilon_F$. When the potential energy is equated with the kinetic energy, p_0 can be defined as below:

$$p_0^2 \approx p_F^2 + p_\lambda^2, \qquad (15)$$

where $p_\lambda = (m\lambda_1 / 2)^{\frac{1}{2}}$. This results in $p_0 > p_F$; hence, the potential energy can be extracted as below:

$$V_1(p_1, p_3) \approx \frac{\lambda_1}{4}, \qquad (16)$$

As a result of applying this potential, the following transition probabilities result:

$$W_{22}(\uparrow\downarrow) = |V_1(p_1, p_3)|^2 \approx \frac{\lambda_1^2}{16}, \qquad (17)$$

$$W_{22}(\uparrow\uparrow) = |V_1(p_1, p_3) - V_1(p_2, p_3)|^2 \approx 0. \qquad (18)$$

In addition, the transition probabilities of the other processes are nearly zero in the limit of low temperatures, and only the binary processes dominate. In this regard, it is emphasized that only the spin-up and spin-down quasiparticles in the binary processes contributed to the transition probabilities at low temperatures. This phenomenon also occurs in superfluid helium three with strong interaction (28). The binary processes are also prevalent in the BEC regime at low temperatures (17).

The superfluid's quasiparticle distribution function, *i.e.*, $\upsilon_{p,\sigma}$ fulfills the kinetic equation as below (29):

$$\frac{\partial \upsilon_{p,\sigma}}{\partial t} + \frac{\partial \upsilon_{p,\sigma}}{\partial \vec{r}} \frac{\partial E_{p,\sigma}}{\partial \vec{p}} - \frac{\partial \upsilon_{p,\sigma}}{\partial \vec{p}} \frac{\partial E_{p,\sigma}}{\partial \vec{r}} = I(\upsilon_{p,\sigma}), \qquad (19)$$

in which $I(\upsilon_{p,\sigma})$ represents the collision integral. A small disturbance of the equilibrium is now considered:

$$\upsilon_\sigma(p,r) = \upsilon^0(p) + \delta\upsilon_\sigma(p), \qquad (20)$$

in general, both $\upsilon(p)$ and $\delta\upsilon_\sigma(p)$ are functions of r, whereas $\upsilon^0(p)$ denotes the equilibrium superfluid distribution function. The function $\psi_\sigma(p)$ is defined as:

$$\delta\upsilon_\sigma(p) = -\frac{1}{T}\upsilon^0(p)(1-\upsilon^0(p))\psi_\sigma(p), \qquad (21)$$

where $\upsilon^0(p) = [\exp(E_p^0 - \vec{p}\cdot\vec{u})/T - 1]^{-1}$, and u denote the slightly inhomogeneous velocity in the gas. The following equation can be obtained by inserting Eq. (20) into Eq. (19), keeping the terms which contribute to the shear viscosity up to the first order in $\delta\upsilon_\sigma(p)$, and assuming $\vec{u}$ and $\vec{\nabla}\cdot\vec{u}$ to be zero at the considered point:

$$-\frac{1}{2}\frac{\partial \upsilon^0}{\partial E_{p,\sigma}} p_i \frac{\partial E_{p,\sigma}}{\partial p_k}\left(\frac{\partial u_i}{\partial r_k} + \frac{\partial u_k}{\partial r_i} - \frac{2}{3}\delta_{ij}\vec{\nabla}\cdot\vec{u}\right) = I(\delta\upsilon_\sigma(p)). \qquad (22)$$

By using Eq. (20) and preserving the first order terms in $\psi_\sigma(p)$, the linearized collision terms in the Boltzmann equation can be formulated as below:

$$I_{22} = \frac{(m^*)^3 T}{64\pi^5 T_F^3}\int W_{22}\frac{\sin\theta}{\cos(\frac{\theta}{2})} d\theta\, d\varphi\, d\varphi_2 \int dx\, dy \big(\psi(t) + \psi(x+y-t) - \psi(x) - \psi(y)\big)$$
$$\times\left[\upsilon^0(t)\upsilon^0(x+y-t)(1-\upsilon^0(x))(1-\upsilon^0(y))\right], \qquad (23)$$

in which φ_2 is the azimuthal angle of $\vec{p}_2$ in relation to $\vec{p}_1$ in addition to $t \equiv E_1 / T$, $x \equiv E_3 / T$, and $y \equiv E_4 / T$.

Moreover, only two quasiparticle processes participate in the collision integral at low temperatures, and only $W_{22}(\uparrow\downarrow)$ is not negligible for this potential model.

The linearized Boltzmann equation can be solved by defining $q(t)$ as below:

$$\psi(t) = p_i \frac{\partial E_1}{\partial p_{1k}} q(t) \left[\frac{\partial u_i}{\partial r_k} + \frac{\partial u_k}{\partial r_i} - \frac{2}{3} \delta_{ij} \vec{\nabla} \cdot \vec{u} \right]. \quad (24)$$

By expanding the bracket in Eq. (24) in a series of the spherical harmonics, *i.e.*, $\sum_{m=-2}^{m=2} U_m \, p_2^{|m|}(\cos\theta) e^{im\Phi}$, and inserting Eq. (24) in the collision integral, the integration on φ_2 can be performed conveniently. Since the collision, integral is symmetric with respect to some variables and integration on y (25), there will be:

$$I_{22} = \frac{(m^*)^3 T^2}{32\pi^4} \sum_{m=-2}^{m=2} U_m \, p_2^{|m|}(\cos\theta) e^{im\Phi} \left(-\frac{\partial \upsilon^0}{\partial E_1} \right) \quad (25)$$
$$\times \int dx \, K(t,x) \int \frac{\sin\theta \, d\theta \, d\varphi}{\cos(\frac{\theta}{2})} W_{22} \left[q(t) + q(-x) p_2(\cos\theta) - q(x)(p_2(\cos\theta_{13}) + p_2(\cos\theta_{14})) \right],$$

where θ_{13} denotes the polar angle between momenta p_1 and p_3, and so on, $p_2^{|m|}$ representing the associated Legendre polynomials, and $K(t,x)$ is then as below:

$$K(t,x) = \frac{2(e-1)}{(e+1)} \frac{e^x}{(e^x+1)} \frac{x-t}{e^{(x-t)}-1}. \quad (26)$$

By inserting Eq. (25) in Eq. (22) and considering $K(t,x) = K(-t,-x)$, the following equation will be derived (25):

$$\int dx \, K(t,x) \left[q_{s_\sigma}(t) - \lambda_{2s_\sigma}(x) \right] = B_\sigma, \quad (27)$$

$$\int dx \, K(t,x) \left[q_{a_\sigma}(t) - \lambda_{2s_\sigma} q_{a_\sigma}(x) \right] = O(TB_\sigma), \quad (28)$$

Where $q_{s_\sigma}(t)$ is the symmetric part, and $q_{a_\sigma}(t)$ is its antisymmetric part of $q(t)$, whereas σ represents the spin index. Eq. (28) is negligible when the temperature is low, whereas Eq. (27) dominates (25). In Eq. (27), λ_{2s_σ} and B_σ are introduced as below:

$$\lambda_{2s_\sigma} = \frac{\int \frac{\sin\theta \, d\theta \, d\varphi}{\cos(\frac{\theta}{2})} W_{22} \left[1 - \frac{3}{4}(1-\cos\theta)^2 \sin^2\varphi \right]}{\int \frac{\sin\theta \, d\theta \, d\varphi}{\cos(\frac{\theta}{2})} W_{22}}, \quad (29)$$

$$B_\sigma = \frac{16\pi^3 k_F^3}{m^{*3} T^2} \frac{(e-1)}{(e+1)} \left[\int \frac{\sin\theta \, d\theta \, d\varphi}{\cos(\frac{\theta}{2})} W_{22} \right]^{-1}, \quad (30)$$

where W_{22} denotes the binary transition probability and does not depend on angles. Generally, for $\sigma = \uparrow$ and $\downarrow$, W_{22} stands for $W_{22}(\uparrow\downarrow)$ and $W_{22}(\downarrow\uparrow)$, respectively. Without a magnetic field, $W_{22}(\uparrow\downarrow) = W_{22}(\downarrow\uparrow)$, with the same function W_{22}. Therefore, W_{22} is equal to $\lambda_1^2/16$ for both cases in Eqs. (29) and (30).

To determine λ_{2s_σ} and B_σ, θ was assumed to be small in the neutral superfluid case, and its maximum value was $\theta_m = \pi T / \Delta(0)$ (30), whereas the maximum gap $\Delta(0)$, for a low temperature fermion gas was $1.76 T_c$ (31). Finally, the following result was achieved:

$$\lambda_{2s_\uparrow} = \lambda_{2s_\downarrow} \approx 1 - \frac{1}{32}\theta_m^4, \quad (31)$$

$$B_{2s_\uparrow} = B_{2s_\downarrow} \approx \frac{16\pi^3 T_F^3}{m^{*3} T^2 W_{22} \theta_m^2} \frac{(e-1)}{(e+1)}. \quad (32)$$

In Sykes and Brooker's procedure (25), Eq. (27) leads to the following output:

$$\int dt \frac{d\upsilon^0}{dt} q_{2s_\sigma}(t) = \frac{-2B(e+1)}{(e-1)(1-\lambda_{2s_\sigma})} c(\lambda_{2s_\sigma}), \quad (33)$$

where $c(\lambda)$ is

$$c(\lambda_{2s_\sigma}) = \frac{(1-\lambda_{2s_\sigma})}{4} \sum_{n=0}^{\infty} \frac{(4n+3)}{(n+1)(2n+1)[(n+1)(2n+1) - \lambda_{2s_\sigma}]} \quad (34)$$
$$= \frac{\lambda_{2s_\sigma} - 1}{2\lambda_{2s_\sigma}} [\gamma + \ln 2 + \frac{1}{2}\psi_d(s_1) + \frac{1}{2}\psi_d(s_2)],$$

where $\gamma = 0.577\cdots$ is Euler's constant, and the digamma function ψ_d is defined by:

$$\psi(z+1) = -\gamma + \sum_{n=1}^{\infty} \frac{z}{n(n+z)}, \quad (35)$$

and

$$s_1 \equiv \frac{3}{4} + \frac{1}{4}\sqrt{8\lambda_{2s_\sigma} + 1} \quad , \quad s_2 \equiv \frac{3}{4} - \frac{1}{4}\sqrt{8\lambda_{2s_\sigma} + 1}. \quad (36)$$

Considering the series expansion in the digamma function and ignoring negligible terms, we concluded $c(\lambda_{2s_\uparrow}) = c(\lambda_{2s_\downarrow}) \approx 0.75$. The value of $c(\lambda_{2s_\sigma})$ was calculated for the superfluid liquid ^{3}He through the Pfitzner procedure (32). It was determined 0.77, which is close to the reported value 0.75. However, the λ and B displayed an entirely different temperature dependance than the λ and B from Shahzamanian and Afzali (28).

The shear viscosity is kept being derived in the next step through the above calculated values of B_σ, λ_{2s_σ}, and $c(\lambda_{2s_\sigma})$.

Defining shear viscosity as a fourth-rank tensor is as follows:

$$\pi_{lm} = -\sum_{ik} \eta_{lmik} \left(\frac{\partial u_i}{\partial r_k} + \frac{\partial u_k}{\partial r_i} - \frac{2}{3}\delta_{ij} \vec{\nabla} \cdot \vec{u} \right), \quad (37)$$

below is the definition of the momentum flux tensor π_{lm}:

$$\pi_{lm} = \int p_l \frac{\partial E}{\partial p_m} \delta\upsilon_\sigma(p)\, d\tau_p \,, \quad (38)$$

after Eq. (24) was inserted in Eq. (21) and then put in Eq. (38), the result was compared with Eq. (37). Then:

$$\eta_{lmik} = -\frac{4p_F^2}{(2\pi)^3 m^*} \int d\Omega_p \hat{p}_l \hat{p}_m \hat{p}_i \hat{p}_k \left[\int dt \frac{\partial \upsilon^0}{\partial t} q_{2s_\uparrow}(t) + \int dt \frac{\partial \upsilon^0}{\partial t} q_{2s_\downarrow} \right]. \quad (39)$$

Eq. (33) was used in Eq. (39), the result was as below:

$$\eta_{lmik} = \frac{p_F^2}{\pi^5 m^*} \int d\Omega_p \hat{p}_l \hat{p}_m \hat{p}_i \hat{p}_k \left(\frac{B_\uparrow}{1-\lambda_{2s_\uparrow}} c(\lambda_{2s_\uparrow}) + \frac{B_\downarrow}{1-\lambda_{2s_\downarrow}} c(\lambda_{2s_\downarrow}) \right), \quad (40)$$

since $B_\uparrow = B_\downarrow$, $\lambda_{2s_\uparrow} = \lambda_{2s_\downarrow}$, and $c(\lambda_{2s_\uparrow}) = c(\lambda_{2s_\downarrow})$ in addition to the fact that the *p*-wave superfluid had two nodes, the following can be written:

$$\eta_{lmik} = \frac{4p_F^2}{\pi^5 m^*} \int d\Omega_p \hat{p}_l \hat{p}_m \hat{p}_i \hat{p}_k \left(\frac{B}{1-\lambda_{2s}} c(\lambda_{2s}) \right). \quad (41)$$

The following result was then obtained after the values of B_σ, λ_{2s_σ}, and $c(\lambda_{2s_\sigma})$ were used, and the angular integrations were determined:

$$\eta_{zz} = 151.30 \frac{p_F^2}{m^{*4} W_{22}} \frac{T_c^4}{T^6}, \quad (42)$$

$$\eta_{xz} = \eta_{yz} = 120.52 \frac{p_F^2}{m^{*4} W_{22} k_B^2} \frac{T_c^2}{T^4}, \quad (43)$$

$$\eta_{xx} = \eta_{yy} = 3\eta_{xy} = 192 \frac{p_F^2}{m^{*4} W_{22}} \frac{1}{T^2}, \quad (44)$$

where each index stands for its double, *i.e.*, η_{xz} means η_{xxzz}. Moreover, the elements of the shear viscosity were determined in the Eqs. (42)-(44). The transition probability W_{22} was written as $\lambda_1^2/16$, where λ_1 indicates the strength of interaction. Furthermore, $p_F = 1.91(N^{1/6}/\bar{a})$, where N represents the number of quasiparticles. Moreover, $\bar{a}$ refers to the characteristic length of the harmonic oscillator, which is $\bar{a} = (\hbar / m\bar{\omega})^{1/2}$, whereas m and $\bar{\omega}$ indicate the atomic mass and the angular frequency, respectively (31). With the values of $\bar{\omega}$ and m from (31) and (33), ^{6}Li and ^{40}K will have the harmonic oscillator lengths of $0.30\times10^{-5}\, m$ and $0.13\times10^{-5}\, m$, respectively. In this case, $N = 10^6$, and ^{6}Li and ^{40}K will have p_F values of $6.72\times10^{-28}\, kgm/s$ and $15.50\times10^{-28}\, kgm/s$, respectively.

The shear viscosity elements in Eqs. (42)-(44) will then be written in terms of the shear viscosity at the temperature T_c, $\eta(T_c)$. Assuming a neutral Fermi gas $\eta(T_c)$ is expressed as below (34):

$$\eta(T_c) = \frac{16}{45T_c^2} \frac{m^* v_F^5}{\left\langle \dfrac{W_{22}\sin^4(\frac{\theta}{2})\sin^2\varphi}{\cos(\frac{\theta}{2})} \right\rangle}, \quad (45)$$

where $W_{22} = W_{22}(\uparrow\downarrow) + \frac{1}{2}W_{22}(\uparrow\uparrow)$ in the normal state when W_{22} is angle-independent. Hence, the following result is concluded:

$$\eta(T_c) = \frac{1}{6\pi} \frac{p_F^5}{m^{*4} W_{22} T_c^2}, \quad (46)$$

where $W_{22} \,\square\, (\lambda_1^2/16)$. After $\eta(T_c)$ is calculated, the elements of shear viscosity are defined as follows:

$$\eta_{zz} = 2851.96\,\eta(T_c)\left(\frac{T_c}{T}\right)^6, \quad (47)$$

$$\eta_{xz} = \eta_{yz} = 2271.74\,\eta(T_c)\left(\frac{T_c}{T}\right)^4, \quad (48)$$

$$\eta_{xx} = \eta_{yy} = 3\eta_{xy} = 3619.14\,\eta(T_c)\left(\frac{T_c}{T}\right)^2. \quad (49)$$

Furthermore, a large part of the temperature dependence of Eqs. (47)-(49) is caused by the values of B and λ in Eqs. (31) and (32).

Results and Discussion

Calculating the transition probabilities of the binary, decay, and coalescence processes was done employing the interaction term in the BCS Hamiltonian. The transition probabilities were subsequently calculated using Nozieres and Schmitt-Rink's generalized separable potential (6). Eventually, the tensor elements of *p*-wave shear viscosity for a Fermi superfluid were calculated. Under the research considerations in this study, the shear viscosity tensor of a uniaxially symmetric system can be expressed as five coefficients of the symmetry axis, $\hat{l}$ (35). Based on the results, only three out of the coefficients were independent.

The shear viscosity of superfluid ^{3}He in the A_1-phase was derived by Roobol *et al.* as $\eta \propto (T)^{-2}$ at low temperatures when only a single spin population was paired (36). They found that in the measurement of the viscosity of η, the η_{zz} values contributed more than the η_{xz} values.

Using the procedures proposed by Pfitzner, Sykes and, Brooker, Shahzamanian and Afzali indicated these temperature dependencies in the superfluid ^3He-A_1 (28). Obviously, the A_1-phase of the superfluid ^3He is composed of the superfluid with spin-up Cooper pairs and the spin-down normal fluid. The shear viscosity elements are temperature dependent through the interaction between these fluids.

Based on the results of this study, η_{xy}, η_{xz}, and η_{zz} depend on temperature as $1/T^2$, $1/T^4$, and $1/T^6$, respectively (refer to Eqs. (42)-(44)). Thermal relaxation rates can be directly determined experimentally, whereas the attenuation of collective modes can be used to extract viscous relaxation rates (10). Regarding viscosity elements, a strong damping was observed in all collective modes in in the *p*-wave state for the ultracold fermion gases and were similar to those of other superfluid Fermi gases (37).

In ultracold fermion gases, the attenuation of collective modes can be measured experimentally to determine whether it is a *s*-wave or a *p*-wave superfluid state. In the *s*-wave state, shear viscosity remains constant at low temperatures (10, 14), while in the *p*-wave state, all elements vary inversely with temperature in powers of 2, 4, and 6.

The bosonic parts of viscosity can be compared to the fermionic parts. The element η_{zz} is proportional to T^{-6} and will dominate at low temperatures gases. In this case, the experimental results provide a useful clue to determine the system state.

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