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RESEARCH PAPER

The study of vibrations in the context of porous micropolar media thermoelasticity and the absence of energy dissipation

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Abstract

In the present article, the theory of linear thermoelasticity without energy dissipation is addressed from the perspective of the analysis of the spatial evolution of harmonic vibrations in time, in the context of a porous micropolar media. Some preliminary identities are determined that will lead to estimates of the harmonic vibration amplitude, some of these estimates being consequences of the distance influence from the disturbed base, provided that a certain critical value for the vibration frequency is considered.

Keywords: micropolar; voids, vibration; thermoelasticity; dissipation

1. Introduction

Here introduce the paper, and put a nomenclature if necessary, with the same font size as the rest of the paper. The paragraphs continue from here and are only separated by headings, subheadings, images and formulae. The section headings are arranged by numbers, bold and 10 pt. Here follows further instructions for authors.

The classical theory of elasticity does not have the possibility of correctly presenting the behaviour of some media that possess an internal structure, see [1-4] for example polycrystalline media or media with fibers, studied by modern engineering. Media that have voids or notches, which constitute stress concentrations, have an apparent strength inversely proportional to the particle size, the greater strength being associated with the smaller-sized particles, exemplified by very thin beames and fibres, for which the bending and torsional forces are greater. Examples of works that address various aspects of these media are [5-14].

The micropolar elasticity is the one suitable for studying these media, for which the classical theory of elasticity is inappropriate, because it takes into account their granularity, see [15-17].

The theory of media with voids has applications in various fields such as the petroleum industry, the pharmaceutical industry, biology and geology.

The starting point of these theories is related to Goodman and Cowin who in [18] introduced an additional degree of kinematic freedom, with the aim of developing the theory of porous fluid media.

The theory of elastic media with voids was extended, in the linear case, by Cowin and Nunziato in [19], where the uniqueness and stability of the weak solution is demonstrated, with the mention that the theory is, in fact, the linear version of the

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nonlinear theory of porous solids, whose behaviour was studied by then in [20]. A study of thermoelastic media, considering the interaction between thermal and mechanical domains, is developed by Ieșan in [21] and extended for porous media in [22].

The theory of thermoelasticity without energy dissipation, whose promoters are Green and Naghdi in [23, 24] is characterized by the appearance of the notion of „thermal displacement”, directly associated with the common temperature and uses the principle introduced by these authors in [25], with reference to a general balance of the entropy, this theory being developed in the circumstances of heat flow in a rigid solid, with specifications for finite speed of the thermal waves propagation.

The linear theory of thermoelasticity without energy dissipation has been the subject of a large number of works, as for example [26, 27] where Chandrasekharaiah formulates, in the context of this theory, the problem with initial and boundary values for isotropic and homogeneous media, [28] where Ciarletta, based on the results obtained by Green and Naghdi, studies the effect of a Galerkin- type solution, determined within a theory of micropolar thermoelasticity, which allows the propagation of thermal waves at finite speed, as well as [29], where Nappa establishes a dynamic principle of Sain-Venant regarding the bounded and the unbounded media respectively, obtaining a uniqueness theorem for unbounded bodies.

Also, in the context of the same linear theory, Ieșan determines in [30] the fundamental solutions, as well as the continuous dependence of the results, Chiriță establishes in [31] a reciprocity theorem for an anisotropic and inhomogeneous media, with a symmetry center at each point, offering an alternative characterization of the mixed problem solution, with initial and boundary data, and Marin and Băleanu study in [32] the vibrations in the thermoelasticity of micropolar media, without energy dissipation. Examples of papers addressing the thermoelasticity without dissipation of energy, also called the type II thermoelasticity, which allows the finite speed propagation of the thermal waves are [30, 33, 34], as well as examples of works that approach the study of vibrations in the context of this type of thermoelasticity, such as [14, 35].

In [36-39], the authors present results regarding the magneto-thermoelastic media.

The structure of this article consists of the formulation, first of all, of the mixed initial boundary value problem, in the case of the porous micropolar media thermoelasticity in the absence of energy dissipation, after which some auxiliary differential identities are demonstrated, these relations constituting the foundation for determining the main result, that of obtaining estimates of the amplitude of harmonic vibration, including those estimates that are consequent of the influence of the distance from the perturbed base, considering a critical value for the vibration frequency.

2. Notations and fundamental equations

It is considered $\mathcal{D}$ an open domain from three - dimensional Euclidean space $\mathbb{R}^3$, to which corresponds, in the reference configuration, a thermoelastic, micropolar, porous, anisotropic and homogeneous media, having the closure and the boundary denoted by $\overline{\mathcal{D}}$, a regular region, respectively $\partial\mathcal{D}$, a smooth surface.

Using a rectangular, fixed Cartesian system of axes $Ox_i, i = \overline{1,3}$, each point of the domain $\mathcal{D}$ is characterized by three orthogonal coordinates, noting that it uses the notation x for (x_1, x_2, x_3) and t for time. In the following, the functions will be regarded as functions of (x, t), defined on the cylinder $\overline{\mathcal{D}} \times (0, \infty)$, where $\overline{\mathcal{D}} = \mathcal{D} \cup \partial\mathcal{D}$. Both the spatial and the time variable arguments will be omitted when the possibility of confusion is excluded. Also, the well-known Einstein summation convention is used, applicable if an index is repeated within a monomial, and the values that the Greek and Latin indices will take are 1,2, respectively 1,2,3.

The partial derivative of a function with respect to time will be denoted by a dot above the function, i.e. $\dot{f} = \frac{\partial f}{\partial t}$, and an index preceded by a comma will represent the partial derivative with respect to the corresponding Cartesian coordinate, i.e. $f_{,j} = \frac{\partial f}{\partial x_j}$.

The independent variables which characterize a porous micropolar thermoelastic media, see **Error! Reference source not found.**, are the displacement vector $\boldsymbol{v} = (v_m)_{1 \le m \le 3}$, the microrotation vector $\boldsymbol{\phi} = (\phi_m)_{1 \le m \le 3}$, the volume function ν corresponding to the pores and θ, the variation of the media temperature compared to the absolute temperature θ_0, which it has in the reference configuration:

$$v_m = v_m(x,t),\ \phi_m = \phi_m(x,t),\ \nu = \nu(x,t),\ \theta = \theta(x,t),\ (x,t) \in \mathcal{D} \times [0,t_0].$$

Following the proposal offered by Green and Naghdi in [23], we will define, with the aid of temperature, the thermal displacement α and the temperature gradient β_i in the form of the following expressions

$$\alpha(x,t) = \int_{t_0}^{t} \theta(x,s)ds,\ \beta_i(x,t) = \int_{t_0}^{t} \theta_{,i}(x,s)ds. \tag{1}$$

Obviously, the relations below are fulfilled

$$\dot{\alpha}(x,t) = \theta(x,t),\ \alpha(x,t_0) = 0,\ \beta_i(x,t) = \alpha_{,i}(x,t) \text{and} \alpha(x,0) = 0,\ \beta_i(x,0) = 0, \tag{2}$$

if $t_0 = 0$.

Representing the strain kinematic characteristics, the tensors $\varepsilon_{ij}, \gamma_{ij}$, as well as the vector v_i are expressed by the geometric equations

$$\begin{aligned} \varepsilon_{ij} &= v_{j,i} + \varepsilon_{jik}\phi_k\ , \\ \gamma_{ij} &= \phi_{j,i}\ , \\ \sigma_i &= \nu_{,i}\ , \end{aligned} \tag{3}$$

where ε_{jik} is the Levi-Civita symbol.

The equations that govern the theory of porous micropolar media thermoelasticity, whithout energy dissipation, are:

- the equations of motion

$$\begin{aligned} t_{ji,j} + \rho f_i &= \rho \ddot{v}_i, \quad \text{in } \mathcal{D} \times (0,\infty), \\ m_{ji,j} + \varepsilon_{ijk}t_{jk} + \rho g_i &= I_{ij}\ddot{\phi}_j, \quad \text{in } \mathcal{D} \times (0,\infty), \end{aligned} \tag{4}$$

- the equilibrated forces balance

$$\lambda_{i,i} + \rho\ell = \rho k \ddot{\nu}, \text{in} \mathcal{D} \times (0,\infty), \tag{5}$$

- the energy equation

$$\rho\dot{\eta} = \frac{\rho}{\theta_0} r - q_{i,i}, \text{in} \mathcal{D} \times (0,\infty). \tag{6}$$

The notations used in the previous equations are presented in the following:

- t_{ji}, m_{ji} are the components of the stress tensor, respectively of the couple stress tensor,
- f_i, g_i are the components of the body force vector, respectively, the components of the couple body force,
- ρ_0 is the mass density in the reference configuration,
- λ_i are the components of the equilibrated stress vector,
- ℓ is the extrinsic equilibrated body force, related to the pores,
- I_{ij} are the microinertia coefficients,
- η is the specific entropy,
- r is the heat supply per unit mass,
- q_i are the components of the heat flux vector,
- k is the inertia coefficient.

Considering that the reference media has a symmetry center at each point, being non-isotropic, the free energy Ψ, through which the constitutive equations are deduced, is given in the form

$$\rho\Psi = \frac{1}{2}C_{mnkl}\varepsilon_{mn}\varepsilon_{kl} + B_{mnkl}\varepsilon_{mn}\gamma_{kl} + \frac{1}{2}A_{mnkl}\gamma_{mn}\gamma_{kl} + B_{mn}\varepsilon_{mn}\nu + C_{mn}\gamma_{mn}\nu + \\ +\frac{1}{2}A_{mn}\sigma_m\sigma_n + \frac{1}{2}g\nu^2 - d_{mn}\varepsilon_{mn}\theta - e_{mn}\gamma_{mn}\theta - m\nu\theta - \frac{c}{2\theta_0}\theta^2 + \frac{c}{2\theta_0}K_{mn}\alpha_{,m}\alpha_{,n}, \tag{7}$$

α representing the thermal displacement related to the variation of temperature, the connection between α and θ being represented by the relation

$$\dot{\alpha}(x,t) = \theta(x,t),\ \alpha(x,0) = 0. \tag{8}$$

At the same time, the coefficients that appear in the form of the free energy ((7)) represent the media characteristics and satisfy the following symmetry relations

$$C_{mnkl} = C_{klmn},\ A_{mnkl} = A_{klmn},\ A_{kl} = A_{lk},\ I_{mn} = I_{nm},\ K_{mn} = K_{nm}, \tag{9}$$

being prescribed functions of class $C^1(\mathcal{D})$.

The constitutive equations, obtained by means of the free energy ((6)), for $(x,t) \in \mathcal{D} \times [0,\infty)$ are

$$\begin{aligned} t_{mn} &= C_{mnkl}\varepsilon_{kl} + B_{mnkl}\gamma_{kl} + B_{mn}\nu - d_{mn}\theta, \\ m_{kl} &= B_{klmn}\varepsilon_{mn} + A_{klmn}\gamma_{mn} + C_{kl}\nu - e_{kl}\theta, \\ \lambda_m &= A_{mn}\nu_{,n}, \\ \rho\eta &= d_{mn}\varepsilon_{mn} + e_{mn}\gamma_{mn} + m\nu + \frac{c}{\theta_0}\theta, \\ q_m &= -\frac{1}{\theta_0}K_{mn}\beta_n, \end{aligned} \tag{10}$$

the system of equations being complete if the heat flow law is added

$$\dot{\beta}_m = \theta_{,m}, \text{ to all } (x,t) \in \bar{\mathcal{D}} \times [0,\infty). \tag{11}$$

Introducing the constitutive equations ((10)) and the geometric equations ((3)) into the motion equations ((4)), the equilibrated forces balance ((5)) and the energy equation ((6)) lead to a system of equations related to the displacements v_m, the microrotations ϕ_m, the volum fraction variation ν and the thermal displacement α, namely

$$\begin{aligned} &[C_{klmn}(v_{n,m} + \varepsilon_{nmr}\phi_r) + B_{klmn}\phi_{n,m} + B_{kl}\nu - d_{kl}\dot{\alpha}]_{,l} + \rho f_k = \rho\ddot{v}_k, \\ &[B_{mnkl}(v_{n,m} + \varepsilon_{nmr}\phi_r) + A_{klmn}\phi_{n,m} + C_{kl}\nu - e_{kl}\dot{\alpha}]_{,l} + \varepsilon_{klr}[C_{lrmn}(v_{n,m} + \varepsilon_{nmr}\phi_r) + \\ &+B_{lrmn}\phi_{n,m} + B_{lr}\nu - d_{lr}\dot{\alpha}] + \rho g_k = I_{kl}\ddot{\varphi}_l, \\ &(A_{mn}\nu_{,n})_{,m} + \rho\ell = \rho k\ddot{\nu}, \\ &\left(\frac{1}{\theta_0}K_{mn}\alpha_{,n}\right)_{,m} - d_{mn}(\dot{v}_{n,m} + \varepsilon_{nmr}\dot{\phi}_r) - e_{mn}\dot{\phi}_{n,m} - m\dot{\nu} + \frac{\rho}{\theta_0}r = \frac{c}{\theta_0}\ddot{\alpha}, \end{aligned} \tag{12}$$

for any $(x,t) \in \mathcal{D} \times (0,\infty)$.

3. Preliminary results

Considering a cross section D of a prismatic cylinder and the section boundary, denoted by ∂D, is presumed to be continuously differetiable. The Cartesian system of orthogonal axes is chosen so that its origin is located at the center of the cylinder base and the positive side of the x_3 axis is conducted along the cylinder.

The length of the cylinder being denoted by L its lateral boundary is $S = \partial D \times [0,L]$. The cylinder content is an anisotropic and homogeneous micropolar media with voids. At the same time, the cylinder is load-free on the lateral surface, so the body force, the couple body force and the flow rate of external heat supply are zero, as well as the displacements, microrotations, the volume fraction variation and the thermal displacement. It is mentioned that on the cylinder base surface the displacements, the microrotations, the volume fraction variation, as well as the thermal displacement are presumed to be harmonic in time. In this context, along with the system of equations ((12)), the following boundary conditions are added for the lateral surface

$$v_m(x,t) = 0,\ \phi_m(x,t) = 0,\ \nu(x,t) = 0,\ \alpha(x,t) = 0,\ \ (x,t) \in S \times (0,\infty), \tag{13}$$

respectively the following boundary conditions for the base

$$\begin{aligned} v_m(x_1,x_2,0,t) &= \tilde{v}_m(x_1,x_2)e^{i\omega t}, \\ \phi_m(x_1,x_2,0,t) &= \tilde{\phi}_m(x_1,x_2)e^{i\omega t}, \\ \nu(x_1,x_2,0,t) &= \tilde{\nu}(x_1,x_2)e^{i\omega t}, \\ \alpha(x_1,x_2,0,t) &= \tilde{\alpha}(x_1,x_2)e^{i\omega t}, \end{aligned} \quad (x_1,x_2) \in D(0), t > 0, \tag{14}$$

where $\tilde{v}_m(x_1,x_2), \tilde{\phi}_m(x_1,x_2), \tilde{\nu}(x_1,x_2)$ and $\tilde{\alpha}(x_1,x_2)$ are smooth, prescribed functions, i is the complex unit and ω is a positive prescribed constant.

The harmonic vibrations in time are generated by the loads in the interior of the cylinder, their form being given by

$$\begin{aligned} v_m(x_1,x_2,x_3,t) &= V_m(x_1,x_2,x_3)e^{i\omega t}, \\ \phi_m(x_1,x_2,x_3,t) &= \Phi_m(x_1,x_2,x_3)e^{i\omega t}, \\ \nu(x_1,x_2,x_3,t) &= \Sigma(x_1,x_2,x_3)e^{i\omega t}, \\ \alpha(x_1,x_2,x_3,t) &= \Lambda(x_1,x_2,x_3)e^{i\omega t}, \end{aligned} \quad (x_1,x_2,x_3,t) \in \mathcal{D} \times (0,\infty). \tag{15}$$

The amplitude $(V_m, \Phi_m, \Sigma, \Lambda)$ of the vibrations satisfies the following system of differential equations:

$$\begin{aligned} &[C_{klmn}(V_{n,m} + \varepsilon_{nmr}\Phi_r) + B_{klmn}\Phi_{n,m} + B_{kl}\Sigma - i\omega d_{kl}\Lambda]_{,l} + \rho\omega^2 V_k = 0, \\ &[B_{mnkl}(V_{n,m} + \varepsilon_{nmr}\Phi_r) + A_{klmn}\Phi_{n,m} + C_{kl}\Sigma - i\omega e_{kl}\Lambda]_{,l} + \varepsilon_{klr}[C_{lrmn}(V_{n,m} + \\ &+\varepsilon_{nmr}\Phi_r) + B_{lrmn}\Phi_{n,m} + B_{lr}\Sigma - i\omega d_{lr}\Lambda] + I_{kl}\omega^2\Phi_l = 0, \\ &(A_{mn}\Sigma_{,n})_{,m} + \rho k\omega^2\Sigma = 0, \\ &\left(\frac{1}{\theta_0}K_{mn}\Lambda_{,n}\right)_{,m} - i\omega d_{mn}(V_{n,m} + \varepsilon_{nmr}\Phi_r) - i\omega e_{mn}\Phi_{n,m} - i\omega m\Sigma + \frac{c}{\theta_0}\omega^2\Lambda = 0. \end{aligned} \tag{16}$$

The boundary conditions of the lateral surface ((13)) take the form

$$V_m(x) = 0, \ \Phi_m(x) = 0, \ \Sigma(x) = 0, \ \Lambda(x) = 0, \ x \in S, \tag{17}$$

and the boundary conditions for the base become

$$\begin{aligned} V_m(x_1,x_2,0) &= \tilde{V}_m(x_1,x_2), \\ \Phi_m(x_1,x_2,0) &= \tilde{\Phi}_m(x_1,x_2), \\ \Sigma(x_1,x_2,0) &= \tilde{\Sigma}(x_1,x_2), \\ \Lambda(x_1,x_2,0) &= \tilde{\Lambda}(x_1,x_2). \end{aligned} \tag{18}$$

In the context of a finite cylinder, the boundary conditions for the upper cylinder base $D(L)$ are prescribed. Regarding an imposed oscillation, the spatial behaviour of the amplitude was studied in [28, 40] as long as the disturbing frequency is lower than a certain critical frequency. The principal purpose of this article is to approximate the evolution of the amplitude in relation to the axial distance from the disturbed base, and in the following, using the procedure presented in [32] the demonstration of some estimates on a solution of the system of differential equations ((16)) is presented, with the boundary conditions for the lateral surface ((17)) and the boundary conditions for the base ((18)). In what follows, the notation

$$\mathcal{V}_{n,m} = V_{n,m} + \varepsilon_{nmr}\Phi_r, \tag{19}$$

will be used.

In the theorem presented below, four identities will be determined that will constitute the foundation for obtaining the principal result.

Theorem 1. *If $(V_m, \Phi_m, \Sigma, \Lambda)$ is a solution of the boundary value problem represented by the equations ((16))-((18)), then the following equalities are fulfilled:*

$$\frac{d}{dx_3}\int_{D(x_3)}\{[C_{3jmn}\mathcal{V}_{n,m}+B_{3jmn}\Phi_{n,m}+B_{3j}\Sigma-i\omega d_{3j}\Lambda]\bar{\mathcal{V}}_j\}dA+$$
$$+\frac{d}{dx_3}\int_{D(x_3)}\{[C_{3jmn}\bar{\mathcal{V}}_{n,m}+B_{3jmn}\bar{\Phi}_{n,m}+B_{3j}\bar{\Sigma}+i\omega d_{3j}\bar{\Lambda}]V_j\}dA+$$
$$+\frac{d}{dx_3}\int_{D(x_3)}\{[B_{3jmn}\mathcal{V}_{n,m}+A_{3jmn}\Phi_{n,m}+C_{3j}\Sigma-i\omega e_{3j}\Lambda]\bar{\Phi}_j\}dA+$$
$$+\frac{d}{dx_3}\int_{D(x_3)}\{[B_{3jmn}\bar{\mathcal{V}}_{n,m}+A_{3jmn}\bar{\Phi}_{n,m}+C_{3j}\bar{\Sigma}+i\omega e_{3j}\bar{\Lambda}]\Phi_j\}dA+$$
$$+\frac{d}{dx_3}\int_{D(x_3)}[(A_{3k}\Sigma_{,k})\bar{\Sigma}+(A_{3k}\bar{\Sigma}_{,k})\Sigma]dA=2\int_{D(x_3)}\{C_{klmn}\mathcal{V}_{l,k}\bar{\mathcal{V}}_{n,m}+ \quad (20)$$
$$+A_{klmn}\Phi_{n,m}\bar{\Phi}_{l,k}+B_{klmn}[\mathcal{V}_{l,k}\bar{\Phi}_{n,m}+\bar{\mathcal{V}}_{l,k}\Phi_{n,m}]+A_{kl}\Sigma_{,k}\bar{\Sigma}_{,l}-\rho\omega^2V_m\bar{V}_m-$$
$$-I_{kl}\omega^2\Phi_k\bar{\Phi}_l-\rho k\omega^2\Sigma\bar{\Sigma}\}dA+\int_{D(x_3)}[B_{kl}(\Sigma\bar{\mathcal{V}}_{l,k}+\bar{\Sigma}\mathcal{V}_{l,k})+$$
$$+C_{kl}(\Sigma\bar{\Phi}_{k,l}+\bar{\Sigma}\Phi_{k,l})+i\omega d_{kl}(\bar{\Lambda}\mathcal{V}_{l,k}-\Lambda\bar{\mathcal{V}}_{l,k})+i\omega e_{kl}(\bar{\Lambda}\Phi_{k,l}-\Lambda\bar{\Phi}_{k,l})]dA;$$

$$\begin{aligned}&\frac{d}{dx_3}\int_{D(x_3)}\{[C_{3jmn}\bar{\mathcal{V}}_{n,m}+B_{3jmn}\bar{\Phi}_{n,m}+B_{3j}\bar{\Sigma}+i\omega d_{3j}\bar{\Lambda}]V_j\}dA-\\&-\frac{d}{dx_3}\int_{D(x_3)}\{[C_{3jmn}\mathcal{V}_{n,m}+B_{3jmn}\Phi_{n,m}+B_{3j}\Sigma-i\omega d_{3j}\Lambda]\bar{V}_j\}dA+\\&+\frac{d}{dx_3}\int_{D(x_3)}\{[B_{3jmn}\bar{\mathcal{V}}_{n,m}+A_{3jmn}\bar{\Phi}_{n,m}+C_{3j}\bar{\Sigma}+i\omega e_{3j}\bar{\Lambda}]\Phi_j\}dA-\\&-\frac{d}{dx_3}\int_{D(x_3)}\{[B_{3jmn}\mathcal{V}_{n,m}+A_{3jmn}\Phi_{n,m}+C_{3j}\Sigma-i\omega e_{3j}\Lambda]\bar{\Phi}_j\}dA+\\&+\frac{d}{dx_3}\int_{D(x_3)}[(A_{3k}\bar{\Sigma}_{,k})\Sigma-(A_{3k}\Sigma_{,k})\bar{\Sigma}]dA=\int_{D(x_3)}[B_{kl}(\bar{\Sigma}\mathcal{V}_{l,k}-\Sigma\bar{\mathcal{V}}_{l,k})+\\&+C_{kl}(\Sigma\bar{\Phi}_{k,l}-\bar{\Sigma}\Phi_{k,l})+i\omega d_{kl}(\bar{\Lambda}\mathcal{V}_{l,k}+\Lambda\bar{\mathcal{V}}_{l,k})+i\omega e_{kl}(\bar{\Lambda}\Phi_{k,l}+\Lambda\bar{\Phi}_{k,l})]dA;\end{aligned} \quad (21)$$

$$\begin{aligned}&\frac{d}{dx_3}\int_{D(x_3)}\frac{1}{\theta_0}K_{3l}(\bar{\Lambda}\Lambda_{,l}+\Lambda\bar{\Lambda}_{,l})dA=2\int_{D(x_3)}\frac{1}{\theta_0}(K_{mn}\Lambda_{,m}\bar{\Lambda}_{,n}-c\omega^2\Lambda\bar{\Lambda})dA+\\&+\int_{D(x_3)}i\omega d_{mn}(\mathcal{V}_{n,m}\bar{\Lambda}-\bar{\mathcal{V}}_{n,m}\Lambda)dA+\int_{D(x_3)}i\omega e_{mn}(\Phi_{n,m}\bar{\Lambda}-\bar{\Phi}_{n,m}\Lambda)dA+\\&+\int_{D(x_3)}i\omega m(\Sigma\bar{\Lambda}-\bar{\Sigma}\Lambda)dA;\end{aligned} \quad (22)$$

$$\begin{aligned}&\frac{d}{dx_3}\int_{D(x_3)}\frac{1}{\theta_0}K_{3l}(\bar{\Lambda}\Lambda_{,l}-\Lambda\bar{\Lambda}_{,l})dA=\int_{D(x_3)}i\omega d_{mn}(\mathcal{V}_{n,m}\bar{\Lambda}+\bar{\mathcal{V}}_{n,m}\Lambda)dA+\\&+\int_{D(x_3)}i\omega e_{mn}(\Phi_{n,m}\bar{\Lambda}+\bar{\Phi}_{n,m}\Lambda)dA+\int_{D(x_3)}i\omega m(\Sigma\bar{\Lambda}+\bar{\Sigma}\Lambda)dA,\end{aligned} \quad (23)$$

where $\bar{z}$ is the complex conjugate of z.

Proof. By multiplying the relations $((16))_1$,$((16))_2$ and $((16))_3$ with $\bar{V}_m$, $\bar{\Phi}_m$, respectively $\bar{\Sigma}$, as well as the conjugates of these relations with V_m, Φ_m and Σ, and then, by adding the six relations obtained, we deduce the equality:

$$\begin{aligned}&\{[C_{ijmn}\mathcal{V}_{n,m}+B_{ijmn}\Phi_{n,m}+B_{ij}\Sigma-i\omega d_{ij}\Lambda]_{,j}+\rho\omega^2V_i\}\bar{V}_i+\\&+\{[C_{ijmn}\bar{\mathcal{V}}_{n,m}+B_{ijmn}\bar{\Phi}_{n,m}+B_{ij}\bar{\Sigma}+i\omega d_{ij}\bar{\Lambda}]_{,j}+\rho\omega^2\bar{V}_i\}V_i+\\&+[B_{mnij}\mathcal{V}_{n,m}+A_{ijmn}\Phi_{n,m}+C_{ij}\Sigma-i\omega e_{ij}\Lambda]_{,j}\bar{\Phi}_i+\\&+[B_{mnij}\bar{\mathcal{V}}_{n,m}+A_{ijmn}\bar{\Phi}_{n,m}+C_{ij}\bar{\Sigma}+i\omega e_{ij}\bar{\Lambda}]_{,j}\Phi_i+\\&+\varepsilon_{ijk}[C_{jkmn}\mathcal{V}_{n,m}+B_{jkmn}\Phi_{n,m}+B_{jk}\Sigma-i\omega d_{jk}\Lambda]\bar{\Phi}_i+I_{mn}\omega^2\bar{\Phi}_m\Phi_n+\\&+\varepsilon_{ijk}[C_{jkmn}\bar{\mathcal{V}}_{n,m}+B_{jkmn}\bar{\Phi}_{n,m}+B_{jk}\bar{\Sigma}+i\omega d_{jk}\bar{\Lambda}]\Phi_i+I_{mn}\omega^2\Phi_m\bar{\Phi}_n+\\&+[(A_{mn}\Sigma_{,n})_{,m}+\rho k\omega^2\Sigma]\bar{\Sigma}+[(A_{mn}\bar{\Sigma}_{,n})_{,m}+\rho k\omega^2\bar{\Sigma}]\Sigma=0.\end{aligned} \quad (24)$$

The previous relation can be rewritten in the following form

$$\begin{aligned}&\{[C_{ijmn}\mathcal{V}_{n,m}+B_{ijmn}\Phi_{n,m}+B_{ij}\Sigma-i\omega d_{ij}\Lambda]\bar{\mathcal{V}}_i\}_{,j}+\{[C_{ijmn}\bar{\mathcal{V}}_{n,m}+B_{ijmn}\bar{\Phi}_{n,m}+\\&+B_{ij}\bar{\Sigma}+i\omega d_{ij}\bar{\Lambda}]\mathcal{V}_i\}_{,j}+\{[B_{mnij}\mathcal{V}_{n,m}+A_{ijmn}\Phi_{n,m}+C_{ij}\Sigma-i\omega e_{ij}\Lambda]\bar{\Phi}_i\}_{,j}+\\&+\{[B_{mnij}\bar{\mathcal{V}}_{n,m}+A_{ijmn}\bar{\Phi}_{n,m}+C_{ij}\bar{\Sigma}+i\omega e_{ij}\bar{\Lambda}]\Phi_i\}_{,j}+[(A_{mn}\Sigma_{,m})\bar{\Sigma}+(A_{mn}\bar{\Sigma}_{,m})\Sigma]_{,n}=\\&2\{C_{ijmn}\mathcal{V}_{j,i}\bar{\mathcal{V}}_{n,m}+A_{ijmn}\Phi_{n,m}\bar{\Phi}_{j,i}+B_{ijmn}[\mathcal{V}_{j,i}\bar{\Phi}_{n,m}+\bar{\mathcal{V}}_{j,i}\Phi_{n,m}]+A_{mn}\Sigma_{,m}\bar{\Sigma}_{,n}-\\&-\rho\omega^2\mathcal{V}_m\bar{\mathcal{V}}_m-I_{mn}\omega^2\Phi_m\bar{\Phi}_n-\rho k\omega^2\Sigma\bar{\Sigma}\}+B_{mn}(\Sigma\bar{\mathcal{V}}_{n,m}+\bar{\Sigma}\mathcal{V}_{n,m})+\\&+C_{mn}(\Sigma\bar{\Phi}_{m,n}+\bar{\Sigma}\Phi_{m,n})+i\omega d_{mn}(\bar{\Lambda}\mathcal{V}_{n,m}-\Lambda\bar{\mathcal{V}}_{n,m})+i\omega e_{mn}(\bar{\Lambda}\Phi_{m,n}-\Lambda\bar{\Phi}_{m,n}).\end{aligned}\tag{25}$$

Integrating the previous equality (25) over $D(x_3)$ and using the lateral surface boundary conditions (17), yields the equality (20).

To prove the relation ((21)), the same relations are considered, i.e. $((16))_1$, $((16))_2$ and $((16))_3$, with the help of which the next equality is reached

$$\begin{aligned}&\{[C_{ijmn}\mathcal{V}_{n,m}+B_{ijmn}\Phi_{n,m}+B_{ij}\Sigma-i\omega d_{ij}\Lambda]_{,j}+\rho\omega^2V_i\}\bar{V}_i-\\&-\{[C_{ijmn}\bar{\mathcal{V}}_{n,m}+B_{ijmn}\bar{\Phi}_{n,m}+B_{ij}\bar{\Sigma}+i\omega d_{ij}\bar{\Lambda}]_{,j}+\rho\omega^2\bar{V}_i\}V_i+\\&+[B_{mnij}\mathcal{V}_{n,m}+A_{ijmn}\Phi_{n,m}+C_{ij}\Sigma-i\omega e_{ij}\Lambda]_{,j}\bar{\Phi}_i-\\&-[B_{mnij}\bar{\mathcal{V}}_{n,m}+A_{ijmn}\bar{\Phi}_{n,m}+C_{ij}\bar{\Sigma}+i\omega e_{ij}\bar{\Lambda}]_{,j}\Phi_i+\\&+\varepsilon_{ijk}[C_{jkmn}\mathcal{V}_{n,m}+B_{jkmn}\Phi_{n,m}+B_{jk}\Sigma-i\omega d_{jk}\Lambda]\bar{\Phi}_i+I_{mn}\omega^2\bar{\Phi}_m\Phi_n-\\&-\varepsilon_{ijk}[C_{jkmn}\bar{\mathcal{V}}_{n,m}+B_{jkmn}\bar{\Phi}_{n,m}+B_{jk}\bar{\Sigma}+i\omega d_{jk}\bar{\Lambda}]\Phi_i-I_{mn}\omega^2\Phi_m\bar{\Phi}_n+\\&+[(A_{mn}\Sigma_{,n})_{,m}+\rho k\omega^2\Sigma]\bar{\Sigma}-[(A_{mn}\bar{\Sigma}_{,n})_{,m}+\rho k\omega^2\bar{\Sigma}]\Sigma=0.\end{aligned}\tag{26}$$

The previous relation ((26)) cam also be written as follows

$$\begin{aligned}&\{[C_{ijmn}\bar{\mathcal{V}}_{n,m}+B_{ijmn}\bar{\Phi}_{n,m}+B_{ij}\bar{\Sigma}+i\omega d_{ij}\bar{\Lambda}]V_i\}_{,j}-\{[C_{ijmn}\mathcal{V}_{n,m}+B_{ijmn}\Phi_{n,m}+\\&+B_{ij}\Sigma-i\omega d_{ij}\Lambda]\bar{V}_i\}_{,j}+\{[B_{ijmn}\bar{\mathcal{V}}_{n,m}+A_{ijmn}\bar{\Phi}_{n,m}+C_{ij}\bar{\Sigma}+i\omega e_{ij}\bar{\Lambda}]\Phi_i\}_{,j}-\\&-\{[B_{ijmn}\mathcal{V}_{n,m}+A_{ijmn}\Phi_{n,m}+C_{ij}\Sigma-i\omega e_{ij}\Lambda]\bar{\Phi}_i\}_{,j}+[(A_{mn}\bar{\Sigma}_{,m})\Sigma-(A_{mn}\Sigma_{,m})\bar{\Sigma}]_{,n}=\\&=B_{mn}(\bar{\Sigma}\mathcal{V}_{n,m}-\Sigma\bar{\mathcal{V}}_{n,m})+C_{mn}(\bar{\Sigma}\Phi_{m,n}-\Sigma\bar{\Phi}_{m,n})+i\omega d_{mn}(\bar{\Lambda}\mathcal{V}_{n,m}+\Lambda\bar{\mathcal{V}}_{n,m})+\\&+i\omega e_{mn}(\bar{\Lambda}\Phi_{m,n}+\Lambda\bar{\Phi}_{m,n}).\end{aligned}\tag{27}$$

By integrating the previous equality (27) on $D(x_3)$ and by applying the boundary conditions ((17)) related to the lateral surface, the equality ((21)) is deduced.

The addition of the relations obtained by multiplying the equation $((16))_4$ by $\bar{\Lambda}$, as well as its conjugate by Λ, lead to the following equality

$$\begin{aligned}&\bar{\Lambda}[(\tfrac{1}{\theta_0}K_{mn}\Lambda_{,n})_{,m}+i\omega d_{mn}\mathcal{V}_{n,m}-i\omega e_{mn}\Phi_{n,m}-i\omega m\Sigma+\tfrac{c}{\theta_0}\omega^2\Lambda]+\\&+\Lambda[(\tfrac{1}{\theta_0}K_{mn}\bar{\Lambda}_{,n})_{,m}+i\omega d_{mn}\bar{\mathcal{V}}_{n,m}+i\omega e_{mn}\bar{\Phi}_{n,m}+i\omega m\bar{\Sigma}+\tfrac{c}{\theta_0}\omega^2\bar{\Lambda}]=0,\end{aligned}\tag{28}$$

equality that can be rewritten in the form

$$\begin{aligned}&\tfrac{2}{\theta_0}(K_{mn}\Lambda_{,n}\bar{\Lambda}_{,m}-c\omega^2\Lambda\bar{\Lambda})+i\omega d_{mn}(\mathcal{V}_{n,m}\bar{\Lambda}-\bar{\mathcal{V}}_{n,m}\Lambda)+i\omega e_{mn}(\Phi_{n,m}\bar{\Lambda}-\bar{\Phi}_{n,m}\Lambda)+\\&+i\omega m(\Sigma\bar{\Lambda}-\bar{\Sigma}\Lambda)=[\tfrac{1}{\theta_0}K_{mn}(\bar{\Lambda}\Lambda_{,n}+\Lambda\bar{\Lambda}_{,n})]_{,m}.\end{aligned}\tag{29}$$

Integrating the relations ((29)) over $D(x_3)$ and using the boundary conditions ((17)) yields the equality ((22)) To demonstrate the last relation ((23)), the same equation $((16))_4$ is used, obtaining the equality

$$\bar{\Lambda}[(\frac{1}{\theta_0}K_{mn}\Lambda_{,n})_{,m} - i\omega d_{mn}\mathcal{V}_{n,m} - i\omega e_{mn}\Phi_{n,m} - i\omega m\Sigma + \frac{c}{\theta_0}\omega^2\Lambda] - \\ -\Lambda[(\frac{1}{\theta_0}K_{mn}\bar{\Lambda}_{,n})_{,m} + i\omega d_{mn}\bar{\mathcal{V}}_{n,m} + i\omega e_{mn}\bar{\Phi}_{n,m} + i\omega m\bar{\Sigma} + \frac{c}{\theta_0}\omega^2\bar{\Lambda}] = 0, \tag{30}$$

which can be presented in the form below

$$i\omega d_{mn}(\mathcal{V}_{n,m}\bar{\Lambda} + \bar{\mathcal{V}}_{n,m}\Lambda) + i\omega e_{mn}(\Phi_{n,m}\bar{\Lambda} + \bar{\Phi}_{n,m}\Lambda) + i\omega m(\Sigma\bar{\Lambda} + \bar{\Sigma}\Lambda) = \\ = [\frac{1}{\theta_0}K_{mn}(\bar{\Lambda}\Lambda_{,n} - \Lambda\bar{\Lambda}_{,n})]_{,m}. \tag{31}$$

Integrating, on $D(x_3)$, the relation above ((31)), and taking into account the boundary conditions ((17)), the relation ((23)) is deduced, which concludes the *proof* of the **Theorem 1**, this being complete. □

The theorem presented in the following is referring equally to the demonstration of some auxiliary identities, necessary to obtain the main result.

Theorem 2. *If $(V_m, \Phi_m, \Sigma, \Lambda)$ is a solution of the boundary value problem expressed through the equations ((16))–((18)), then the following equalities are satisfied:*

$$\int_{D(x_3)} [C_{ijmn}\mathcal{V}_{n,m}\bar{\mathcal{V}}_{j,i} + A_{ijmn}\Phi_{i,j}\bar{\Phi}_{n,m} + A_{mn}\Sigma_{,m}\bar{\Sigma}_{,n}]dA + \int_{D(x_3)} [B_{ijmn}(\mathcal{V}_{n,m}\bar{\Phi}_{i,j} + \\ +\bar{\mathcal{V}}_{n,m}\Phi_{i,j}) - 3\omega^2(\rho V_m\bar{V}_m + \rho k\Sigma\bar{\Sigma} + I_{mn}\Phi_m\bar{\Phi}_n)]dA + 2\int_{D(x_3)} [B_{mn}(\mathcal{V}_{n,m}\bar{\Sigma} + \\ +\bar{\mathcal{V}}_{n,m}\Sigma) + C_{mn}(\Phi_{m,n}\bar{\Sigma} + \bar{\Phi}_{m,n}\Sigma)]dA - 2i\omega\int_{D(x_3)} [d_{mn}(\Lambda\bar{\mathcal{V}}_{n,m} - \bar{\Lambda}\mathcal{V}_{n,m}) + e_{mn}(\Lambda\bar{\Phi}_{n,m} - \\ -\bar{\Lambda}\Phi_{n,m})]dA + \int_{D(x_3)} [x_pB_{mn}(\Sigma_{,p}\bar{\mathcal{V}}_{n,m} + \bar{\Sigma}_{,p}\mathcal{V}_{n,m}) + x_pC_{mn}(\Sigma_{,p}\bar{\Phi}_{m,n} + \bar{\Sigma}_{,p}\Phi_{m,n})]dA - \\ -i\omega\int_{D(x_3)} [d_{mn}x_p(\Lambda_{,p}\bar{\mathcal{V}}_{n,m} - \bar{\Lambda}_{,p}\mathcal{V}_{n,m}) + e_{mn}x_p(\Lambda_{,p}\bar{\Phi}_{n,m} - \bar{\Lambda}_{,p}\Phi_{n,m})]dA = \\ -\frac{d}{dx_3}\int_{D(x_3)} \{[C_{3jmn}\mathcal{V}_{n,m} + B_{3jmn}\Phi_{n,m} + B_{3j}\Sigma - i\omega d_{3j}\Lambda]x_p\bar{V}_{j,p}\}dA - \\ -\frac{d}{dx_3}\int_{D(x_3)} \{[C_{3jmn}\bar{\mathcal{V}}_{n,m} + B_{3jmn}\bar{\Phi}_{n,m} + B_{3j}\bar{\Sigma} + i\omega d_{3j}\bar{\Lambda}]x_pV_{j,p}\}dA - \\ -\frac{d}{dx_3}\int_{D(x_3)} \{[B_{3jmn}\mathcal{V}_{n,m} + A_{3jmn}\Phi_{n,m} + C_{3j}\Sigma - i\omega e_{3j}\Lambda]x_p\bar{\Phi}_{j,p}\}dA - \\ -\frac{d}{dx_3}\int_{D(x_3)} \{[B_{3jmn}\bar{\mathcal{V}}_{n,m} + A_{3jmn}\bar{\Phi}_{n,m} + C_{3j}\bar{\Sigma} + i\omega e_{3j}\bar{\Lambda}]x_p\Phi_{j,p}\}dA - \\ -\frac{d}{dx_3}\int_{D(x_3)} [(A_{3j}\Sigma_{,j})x_p\bar{\Sigma}_{,p} + (A_{3j}\bar{\Sigma}_{,j})x_p\Sigma_{,p}]dA + \frac{d}{dx_3}\int_{D(x_3)} x_3[C_{ijmn}\mathcal{V}_{n,m}\bar{\mathcal{V}}_{j,i} + \\ +A_{ijmn}\Phi_{i,j}\bar{\Phi}_{n,m} + A_{mn}\Sigma_{,m}\bar{\Sigma}_{,n}]dA + \frac{d}{dx_3}\int_{D(x_3)} x_3\{B_{ijmn}[\mathcal{V}_{n,m}\bar{\Phi}_{i,j} + \bar{\mathcal{V}}_{n,m}\Phi_{i,j}] + \\ +B_{mn}[\mathcal{V}_{n,m}\bar{\Sigma} + \bar{\mathcal{V}}_{n,m}\Sigma] + C_{mn}[\bar{\Sigma}\Phi_{m,n} + \Sigma\bar{\Phi}_{m,n}] - \rho\omega^2V_m\bar{V}_m - \rho k\omega^2\Sigma\bar{\Sigma}\}dA - \\ -\frac{d}{dx_3}\int_{D(x_3)} i\omega x_3[d_{mn}(\Lambda\bar{\mathcal{V}}_{n,m} - \bar{\Lambda}\mathcal{V}_{n,m}) + e_{mn}(\Lambda\bar{\Phi}_{n,m} - \bar{\Lambda}\Phi_{n,m}) - x_3I_{mn}\omega^2\Phi_m\bar{\Phi}_n]dA - \\ -\int_{\partial D(x_3)} x_pn_p(C_{i\alpha m\beta}n_\alpha n_\beta\frac{\partial\mathcal{V}_i}{\partial n}\frac{\partial\bar{\mathcal{V}}_m}{\partial n}) + 2B_{i\alpha m\beta}n_\alpha n_\beta\frac{\partial\mathcal{V}_i}{\partial n}\frac{\partial\Phi_m}{\partial n} + \\ +A_{i\alpha m\beta}n_\alpha n_\beta\frac{\partial\Phi_i}{\partial n}\frac{\partial\bar{\Phi}_m}{\partial n} + A_{\alpha\beta}n_\alpha n_\beta\frac{\partial\Sigma}{\partial n}\frac{\partial\bar{\Sigma}}{\partial n})ds; \tag{32}$$

$$\int_{D(x_3)} \frac{1}{\theta_0}(K_{mn}\Lambda_{,m}\bar{\Lambda}_{,n} - 3c\omega\Lambda\bar{\Lambda})dA + \int_{D(x_3)} i\omega d_{mn}x_p(\bar{\mathcal{V}}_{n,m}\Lambda_{,p} - \mathcal{V}_{n,m}\bar{\Lambda}_{,p})dA + \\ +\int_{D(x_3)} i\omega e_{mn}x_p(\bar{\Phi}_{n,m}\Lambda_{,p} - \Phi_{n,m}\bar{\Lambda}_{,p})dA + \int_{D(x_3)} i\omega mx_p(\bar{\Sigma}\Lambda_{,p} - \Sigma\bar{\Lambda}_{,p})dA + \\ +\int_{\partial D(x_3)} x_pn_pK_{\alpha\beta}n_\alpha n_\beta\frac{\partial\Lambda}{\partial n}\frac{\partial\bar{\Lambda}}{\partial n}ds = -\frac{d}{dx_3}\int_{D(x_3)} \frac{1}{\theta_0}[x_\alpha K_{3\beta}(\bar{\Lambda}_{,\alpha}\Lambda_{,\beta} + \Lambda_{,\alpha}\bar{\Lambda}_{,\beta}) + \\ +x_\alpha K_{33}(\bar{\Lambda}_{,\alpha}\Lambda_{,3} + \Lambda_{,\alpha}\bar{\Lambda}_{,3})] - \frac{d}{dx_3}\int_D \frac{x_3}{\theta_0}(K_{33}\Lambda_{,3}\bar{\Lambda}_{,3} - K_{\alpha\beta}\Lambda_{,\alpha}\bar{\Lambda}_{,\beta} + c\omega^2\Lambda\bar{\Lambda})dA. \tag{33}$$

Proof. Using the equation $((16))_1$–$((16))_3$, the following equality holds

$$\{[C_{ijmn}\mathcal{V}_{n,m}+B_{ijmn}\Phi_{n,m}+B_{ij}\Sigma-i\omega d_{ij}\Lambda]_{,j}+\rho\omega^2V_i\}x_p\bar{V}_{i,p}+[B_{mnij}\mathcal{V}_{n,m}+ +A_{ijmn}\Phi_{n,m}+C_{ij}\Sigma-i\omega e_{ij}\Lambda]_{,j}x_p\bar{\Phi}_{i,p}+\varepsilon_{ijk}[C_{jkmn}\mathcal{V}_{n,m}+B_{jkmn}\Phi_{n,m}+B_{jk}\Sigma- -i\omega d_{jk}\Lambda]x_p\bar{\Phi}_{i,p}+I_{mn}\omega^2x_p\Phi_n\bar{\Phi}_{m,p}+[(A_{mn}\Sigma_{,n})_{,m}+\rho k\omega^2\Sigma]x_p\bar{\Sigma}_{,p}+\{[C_{ijmn}\bar{\mathcal{V}}_{n,m}+ +B_{ijmn}\bar{\Phi}_{n,m}+B_{ij}\bar{\Sigma}+i\omega d_{ij}\bar{\Lambda}]_{,j}+\rho\omega^2\bar{V}_i\}x_pV_{i,p}+[B_{mnij}\bar{\mathcal{V}}_{n,m}+A_{ijmn}\bar{\Phi}_{n,m}+ +C_{ij}\bar{\Sigma}+i\omega e_{ij}\bar{\Lambda}]_{,j}x_p\Phi_{i,p}+\varepsilon_{ijk}[C_{jkmn}\bar{\mathcal{V}}_{n,m}+B_{jkmn}\bar{\Phi}_{n,m}+B_{jk}\bar{\Sigma}+i\omega d_{jk}\bar{\Lambda}]x_p\Phi_{i,p}+ +I_{mn}\omega^2x_p\bar{\Phi}_n\Phi_{m,p}+[(A_{mn}\bar{\Sigma}_{,n})_{,m}+\rho k\omega^2\bar{\Sigma}]x_p\Sigma_{,p}=0. \tag{34}$$

The previous relation can also be written in the form

$$\begin{aligned}
&\{[C_{ijmn}\mathcal{V}_{n,m}+B_{ijmn}\Phi_{n,m}+B_{ij}\Sigma-i\omega d_{ij}\Lambda]x_p\bar{V}_{i,p}\}_{,j}-\\
&-[C_{ijmn}\mathcal{V}_{n,m}+B_{ijmn}\Phi_{n,m}+B_{ij}\Sigma-i\omega d_{ij}\Lambda]x_p\bar{V}_{i,pj}+\rho\omega^2V_mx_p\bar{V}_{m,p}+\\
&+\{[B_{mnij}\mathcal{V}_{n,m}+A_{ijmn}\Phi_{n,m}+C_{ij}\Sigma-i\omega e_{ij}\Lambda]x_p\bar{\Phi}_{i,p}\}_{,j}-\\
&-[B_{mnij}\mathcal{V}_{n,m}+A_{ijmn}\Phi_{n,m}+C_{ij}\Sigma-i\omega e_{ij}\Lambda]x_p\bar{\Phi}_{i,pj}+\\
&+\varepsilon_{ijk}[C_{jkmn}\mathcal{V}_{n,m}+B_{jkmn}\Phi_{n,m}+B_{jk}\Sigma-i\omega d_{jk}\Lambda]x_p\bar{\Phi}_{i,p}+\\
&+I_{mn}\omega^2x_p\Phi_m\bar{\Phi}_{n,p}+[(A_{mn}\Sigma_{,m})x_p\bar{\Sigma}_{,p}]_{,n}-(A_{mn}\Sigma_{,m})x_p\bar{\Sigma}_{,pn}+\rho k\omega^2x_p\Sigma\bar{\Sigma}_{,p}+\\
&+\{[C_{ijmn}\bar{\mathcal{V}}_{n,m}+B_{ijmn}\bar{\Phi}_{n,m}+B_{ij}\bar{\Sigma}+i\omega d_{ij}\bar{\Lambda}]x_pV_{i,p}\}_{,j}-\\
&-[C_{ijmn}\bar{\mathcal{V}}_{n,m}+B_{ijmn}\bar{\Phi}_{n,m}+B_{ij}\bar{\Sigma}+i\omega d_{ij}\bar{\Lambda}]x_pV_{i,pj}+\\
&+\rho\omega^2\bar{V}_mx_pV_{m,p}+\{[B_{mnij}\bar{\mathcal{V}}_{n,m}+A_{ijmn}\bar{\Phi}_{n,m}+C_{ij}\bar{\Sigma}+i\omega e_{ij}\bar{\Lambda}]x_p\Phi_{i,p}\}_{,j}-\\
&-[B_{mnij}\bar{\mathcal{V}}_{n,m}+A_{ijmn}\bar{\Phi}_{n,m}+C_{ij}\bar{\Sigma}+i\omega e_{ij}\bar{\Lambda}]x_p\Phi_{i,pj}+\\
&+\varepsilon_{ijk}[C_{jkmn}\bar{\mathcal{V}}_{n,m}+B_{jkmn}\bar{\Phi}_{n,m}+B_{jk}\bar{\Sigma}+i\omega d_{jk}\bar{\Lambda}]x_p\Phi_{i,p}+\\
&+I_{mn}\omega^2x_p\bar{\Phi}_n\Phi_{m,p}+[(A_{mn}\bar{\Sigma}_{,m})x_p\Sigma_{,p}]_{,n}-(A_{mn}\bar{\Sigma}_{,m})x_p\Sigma_{,pn}+\rho k\omega^2x_p\bar{\Sigma}\Sigma_{,p}=0.
\end{aligned} \tag{35}$$

The above equality (35) leads to the form

$$\begin{aligned}
&C_{ijmn}\mathcal{V}_{n,m}\bar{\mathcal{V}}_{j,i}+A_{ijmn}\Phi_{i,j}\bar{\Phi}_{n,m}+A_{mn}\Sigma_{,m}\bar{\Sigma}_{,n}+\\
&+B_{ijmn}(\mathcal{V}_{n,m}\bar{\Phi}_{i,j}+\bar{\mathcal{V}}_{n,m}\Phi_{i,j})-3\omega^2(\rho V_m\bar{V}_m+\rho k\Sigma\bar{\Sigma}+I_{mn}\Phi_m\bar{\Phi}_n)+\\
&+2B_{mn}(\mathcal{V}_{n,m}\bar{\Sigma}+\bar{\mathcal{V}}_{n,m}\Sigma)+2C_{mn}(\Phi_{m,n}\bar{\Sigma}+\bar{\Phi}_{m,n}\Sigma)-\\
&-2i\omega d_{mn}(\Lambda\bar{\mathcal{V}}_{n,m}-\bar{\Lambda}\mathcal{V}_{n,m})-2i\omega e_{mn}(\Lambda\bar{\Phi}_{n,m}-\bar{\Lambda}\Phi_{n,m})+\\
&+x_pB_{mn}(\Sigma_{,p}\bar{\mathcal{V}}_{n,m}+\bar{\Sigma}_{,p}\mathcal{V}_{n,m})+x_pC_{mn}(\Sigma_{,p}\bar{\Phi}_{m,n}+\bar{\Sigma}_{,p}\Phi_{m,n})-\\
&-i\omega d_{mn}x_p(\Lambda_{,p}\bar{\mathcal{V}}_{n,m}-\bar{\Lambda}_{,p}\mathcal{V}_{n,m})-i\omega e_{mn}x_p(\Lambda_{,p}\bar{\Phi}_{n,m}-\bar{\Lambda}_{,p}\Phi_{n,m})=\\
&=-\{[C_{ijmn}\mathcal{V}_{n,m}+B_{ijmn}\Phi_{n,m}+B_{ij}\Sigma-i\omega d_{ij}\Lambda]x_p\bar{V}_{i,p}\}_{,j}-\\
&-\{[C_{ijmn}\bar{\mathcal{V}}_{n,m}+B_{ijmn}\bar{\Phi}_{n,m}+B_{ij}\bar{\Sigma}+i\omega d_{ij}\bar{\Lambda}]x_pV_{i,p}\}_{,j}-\\
&-\{[B_{mnij}\mathcal{V}_{n,m}+A_{ijmn}\Phi_{n,m}+C_{ij}\Sigma-i\omega e_{ij}\Lambda]x_p\bar{\Phi}_{i,p}\}_{,j}-\\
&-\{[B_{mnij}\bar{\mathcal{V}}_{n,m}+A_{ijmn}\bar{\Phi}_{n,m}+C_{ij}\bar{\Sigma}+i\omega e_{ij}\bar{\Lambda}]x_p\Phi_{i,p}\}_{,j}-\\
&-[(A_{mn}\Sigma_{,m})x_p\bar{\Sigma}_{,p}]_{,n}-[(A_{mn}\bar{\Sigma}_{,m})x_p\Sigma_{,p}]_{,n}+[x_pC_{ijmn}\mathcal{V}_{n,m}\bar{\mathcal{V}}_{j,i}+\\
&+x_pA_{ijmn}\Phi_{i,j}\bar{\Phi}_{n,m}+x_pA_{mn}\Sigma_{,m}\bar{\Sigma}_{,n}]_{,p}+\{x_pB_{ijmn}[\mathcal{V}_{n,m}\bar{\Phi}_{i,j}+\bar{\mathcal{V}}_{n,m}\Phi_{i,j}]+
\end{aligned} \tag{36}$$

$$+x_pB_{mn}(\mathcal{V}_{n,m}\bar{\Sigma}+\bar{\mathcal{V}}_{n,m}\Sigma)+x_pC_{mn}(\bar{\Sigma}\Phi_{m,n}+\Sigma\bar{\Phi}_{m,n})-x_p\rho\omega^2V_m\bar{V}_m-x_p\rho k\omega^2\Sigma\bar{\Sigma}\}_{,p}-$$
$$-\{i\omega x_pd_{mn}(\Lambda\bar{\mathcal{V}}_{n,m}-\bar{\Lambda}\mathcal{V}_{n,m})\}_{,p}-\{i\omega x_pe_{mn}(\Lambda\bar{\Phi}_{n,m}-\bar{\Lambda}\Phi_{n,m})-x_pI_{mn}\omega^2\Phi_m\bar{\Phi}_n\}_{,p}.$$

By integrating the previous equality (36) over $D(x_3)$ and taking into consideration the boundary conditions ((17)), the next identity is obtained:

$$\int_{D(x_3)}[C_{ijmn}\mathcal{V}_{n,m}\bar{\mathcal{V}}_{j,i}+A_{ijmn}\Phi_{i,j}\bar{\Phi}_{n,m}+A_{mn}\Sigma_{,m}\bar{\Sigma}_{,n}]dA+$$
$$+\int_{D(x_3)}[B_{ijmn}(\mathcal{V}_{n,m}\bar{\Phi}_{i,j}+\bar{\mathcal{V}}_{n,m}\Phi_{i,j})-3\omega^2(\rho V_m\bar{V}_m+\rho k\Sigma\bar{\Sigma}+I_{mn}\Phi_m\Phi_n)]dA+$$
$$+2\int_{D(x_3)}[B_{mn}(\mathcal{V}_{n,m}\bar{\Sigma}+\bar{\mathcal{V}}_{n,m}\Sigma)+C_{mn}(\Phi_{m,n}\bar{\Sigma}+\bar{\Phi}_{m,n}\Sigma)]dA-$$
$$-2i\omega\int_{D(x_3)}[d_{mn}(\Lambda\bar{\mathcal{V}}_{n,m}-\bar{\Lambda}\mathcal{V}_{n,m})+e_{mn}(\Lambda\bar{\Phi}_{n,m}-\bar{\Lambda}\Phi_{n,m})]dA+$$
$$+\int_{D(x_3)}[x_pB_{mn}(\Sigma_{,p}\bar{\mathcal{V}}_{n,m}+\Sigma_{,p}\mathcal{V}_{n,m})+x_pC_{mn}(\Sigma_{,p}\bar{\Phi}_{m,n}+\bar{\Sigma}_{,p}\Phi_{m,n})]dA-$$
$$-i\omega\int_{D(x_3)}[d_{mn}x_p(\Lambda_{,p}\bar{\mathcal{V}}_{n,m}-\bar{\Lambda}_{,p}\mathcal{V}_{n,m})+e_{mn}x_p(\Lambda_{,p}\bar{\Phi}_{n,m}-\bar{\Lambda}_{,p}\Phi_{n,m})]dA=$$
$$-\frac{d}{dx_3}\int_{D(x_3)}\{[C_{3jmn}\mathcal{V}_{n,m}+B_{3jmn}\Phi_{n,m}+B_{3j}\Sigma-i\omega d_{3j}\Lambda]x_p\bar{V}_{j,p}\}dA-$$
$$-\frac{d}{dx_3}\int_{D(x_3)}\{[C_{3jmn}\bar{\mathcal{V}}_{n,m}+B_{3jmn}\bar{\Phi}_{n,m}+B_{3j}\bar{\Sigma}+i\omega d_{3j}\bar{\Lambda}]x_pV_{j,p}\}dA-$$
$$-\frac{d}{dx_3}\int_{D(x_3)}\{[B_{3jmn}\mathcal{V}_{n,m}+A_{3jmn}\Phi_{n,m}+C_{3j}\Sigma-i\omega e_{3j}\Lambda]x_p\bar{\Phi}_{j,p}\}dA-$$
$$-\frac{d}{dx_3}\int_{D(x_3)}\{[B_{3jmn}\bar{\mathcal{V}}_{n,m}+A_{3jmn}\bar{\Phi}_{n,m}+C_{3j}\bar{\Sigma}+i\omega e_{3j}\bar{\Lambda}]x_p\bar{\Phi}_{j,p}\}dA- \quad (37)$$
$$-\frac{d}{dx_3}\int_{D(x_3)}[(A_{3j}\Sigma_{,j})x_p\bar{\Sigma}_{,p}+(A_{3j}\bar{\Sigma}_{,j})x_p\Sigma_{,p}]dA+\frac{d}{dx_3}\int_{D(x_3)}x_3[C_{ijmn}\mathcal{V}_{n,m}\bar{\mathcal{V}}_{j,i}+$$
$$+A_{ijmn}\Phi_{i,j}\bar{\Phi}_{n,m}+A_{mn}\Sigma_{,m}\bar{\Sigma}_{,n}]dA+\frac{d}{dx_3}\int_{D(x_3)}x_3\{B_{ijmn}[\mathcal{V}_{n,m}\bar{\Phi}_{i,j}+\bar{\mathcal{V}}_{n,m}\Phi_{i,j}]+$$
$$+B_{mn}[\mathcal{V}_{n,m}\bar{\Sigma}+\bar{\mathcal{V}}_{n,m}\Sigma]+C_{mn}[\bar{\Sigma}\Phi_{m,n}+\Sigma\bar{\Phi}_{m,n}]-\rho\omega^2V_m\bar{V}_m-\rho k\omega^2\Sigma\bar{\Sigma}\}dA-$$
$$-\frac{d}{dx_3}\int_{D(x_3)}i\omega x_3[d_{mn}(\Lambda\bar{\mathcal{V}}_{n,m}-\bar{\Lambda}\mathcal{V}_{n,m})+e_{mn}(\Lambda\bar{\Phi}_{n,m}-\bar{\Lambda}\Phi_{n,m})-x_3I_{mn}\omega^2\Phi_m\bar{\Phi}_n]dA-$$
$$-\int_{\partial D(x_3)}[x_pC_{psmn}\mathcal{V}_{n,m}\bar{\mathcal{V}}_{s,p}+x_pC_{psmn}\bar{\mathcal{V}}_{n,m}\mathcal{V}_{s,p}]n_pds-2\int_{\partial D(x_3)}[x_pB_{psmn}\mathcal{V}_{n,m}\bar{\Phi}_{s,p}$$
$$+x_pB_{psmn}\bar{\mathcal{V}}_{n,m}\Phi_{s,p}]n_pds-\int_{\partial D(x_3)}[x_pA_{psmn}\Phi_{n,m}\bar{\Phi}_{s,p}+x_pA_{psmn}\bar{\Phi}_{n,m}\Phi_{s,p}]n_pds-$$
$$-\int_{\partial D(x_3)}[x_pA_{ps}\Sigma_{,p}\bar{\Sigma}_{,s}+x_pA_{ps}\bar{\Sigma}_{,p}\Sigma_{,s}]n_pds+\int_{\partial D(x_3)}x_pn_p\{C_{ijmn}\bar{\mathcal{V}}_{n,m}\mathcal{V}_{n,m}+$$
$$+B_{ijmn}[\mathcal{V}_{j,i}\bar{\Phi}_{n,m}+\bar{\mathcal{V}}_{j,i}\Phi_{n,m}]+A_{ijmn}\Phi_{j,i}\bar{\Phi}_{n,m}+A_{mn}\Sigma_{,m}\bar{\Sigma}_{,n}\}ds.$$

Considering the boundary conditions ((17)), corresponding to the lateral surface, it is deduced that

$$V_{i,3}=0,\ \text{pe}\ \ \partial D(x_3). \quad (38)$$

On the curve ∂D, the following relation is satisfied

$$V_{m,\alpha}=n_\alpha\frac{\partial V_m}{\partial n}+\tau_\alpha\frac{\partial V_m}{\partial\tau}, \quad (39)$$

where τ_α represent the components of the unit tangent vector to ∂D. Under the influence of the conditions ((17)), it is deduced that $\frac{\partial V_m}{\partial\tau}=0$ on the curve ∂D, so the following relations are obtained

$$V_{m,\alpha} = n_\alpha \frac{\partial V_m}{\partial n}, \text{onthecurve } \partial D, \tag{40}$$

and

$$\Sigma_{,\alpha} = n_\alpha \frac{\partial \Sigma}{\partial n}, \text{ on the curve } \partial D. \tag{41}$$

Using the equations ((38)),((40)) and ((41)), the last integral of the equality (38) becomes

$$\begin{aligned} &\int_{\partial D(x_3)} x_p n_p \{C_{ijmn} \bar{\mathcal{V}}_{n,m} \mathcal{V}_{j,i} + B_{ijmn}[\mathcal{V}_{j,i} \bar{\Phi}_{n,m} + \bar{\mathcal{V}}_{j,i} \Phi_{n,m}] + A_{ijmn} \Phi_{j,i} \bar{\Phi}_{n,m} + \\ &+ A_{mn} \Sigma_{,m} \bar{\Sigma}_{,n}\} ds = \int_{\partial D(x_3)} x_p n_p (C_{i\alpha m\beta} n_\alpha n_\beta \frac{\partial \mathcal{V}_i}{\partial n} \frac{\partial \bar{\mathcal{V}}_m}{\partial n} + 2 B_{i\alpha m\beta} n_\alpha n_\beta \frac{\partial \mathcal{V}_i}{\partial n} \frac{\partial \bar{\Phi}_m}{\partial n} + \\ &+ A_{i\alpha m\beta} n_\alpha n_\beta \frac{\partial \Phi_i}{\partial n} \frac{\partial \bar{\Phi}_m}{\partial n} + A_{\alpha\beta} n_\alpha n_\beta \frac{\partial \Sigma}{\partial n} \frac{\partial \bar{\Sigma}}{\partial n}) ds. \end{aligned} \tag{42}$$

For the other integrals of this type from the relation (38), the next equalities are obtained

$$\begin{aligned} &\int_{\partial D(x_3)} [x_p C_{psmn} \mathcal{V}_{n,m} \bar{\mathcal{V}}_{s,p} + x_p C_{psmn} \bar{\mathcal{V}}_{n,m} \mathcal{V}_{s,p}] n_p ds = \\ &= 2 \int_{\partial D(x_3)} x_p n_p C_{i\alpha m\beta} n_\alpha n_\beta \frac{\partial \mathcal{V}_i}{\partial n} \frac{\partial \bar{\mathcal{V}}_m}{\partial n} ds, \\ &\int_{\partial D(x_3)} [x_p B_{psmn} \mathcal{V}_{n,m} \bar{\Phi}_{s,p} + x_p A_{psmn} \bar{\mathcal{V}}_{n,m} \Phi_{s,p}] n_p ds = \\ &= 2 \int_{\partial D(x_3)} x_p n_p B_{i\alpha m\beta} n_\alpha n_\beta \frac{\partial \mathcal{V}_i}{\partial n} \frac{\partial \bar{\Phi}_m}{\partial n} ds, \\ &\int_{\partial D(x_3)} [x_p A_{psmn} \Phi_{n,m} \bar{\Phi}_{s,p} + x_p A_{psmn} \bar{\Phi}_{n,m} \Phi_{s,p}] n_p ds = \\ &= 2 \int_{\partial D(x_3)} x_p n_p A_{i\alpha m\beta} n_\alpha n_\beta \frac{\partial \Phi_i}{\partial n} \frac{\partial \bar{\Phi}_m}{\partial n} ds, \\ &\int_{\partial D(x_3)} [x_p A_{ps} \Sigma_{,p} \bar{\Sigma}_{,s} + x_p A_{ps} \bar{\Sigma}_{,p} \Sigma_{,s}] n_p ds = 2 \int_{\partial D(x_3)} x_p n_p A_{\alpha\beta} \frac{\partial \Sigma}{\partial n} \frac{\partial \bar{\Sigma}}{\partial n} ds. \end{aligned} \tag{43}$$

Replacing the relations ((42)) and (43) in the equality (37) leads to obtaining the first equation of the Theorem 2, denoted by (32).

In order to demonstrate the relation ((33)), we use the equality $((16))_3$ and deduce the following relation

$$\begin{aligned} &x_p \bar{\Lambda}_{,p} [(\frac{1}{\theta_0} K_{mn} \Lambda_{,n})_{,m} - i\omega d_{mn} \mathcal{V}_{n,m} - i\omega e_{mn} \Phi_{n,m} - i\omega m \Sigma + \frac{c}{\theta_0} \omega^2 \Lambda] + \\ &+ x_p \Lambda_{,p} [(\frac{1}{\theta_0} K_{mn} \bar{\Lambda}_{,n})_{,m} + i\omega d_{mn} \bar{\mathcal{V}}_{n,m} + i\omega e_{mn} \bar{\Phi}_{n,m} + i\omega m \bar{\Sigma} + \frac{c}{\theta_0} \omega^2 \bar{\Lambda}] = 0. \end{aligned} \tag{44}$$

To above equality can also be written in the form

$$\begin{aligned} &i\omega d_{mn} x_p (\bar{\mathcal{V}}_{n,m} \Lambda_{,p} - \mathcal{V}_{n,m} \bar{\Lambda}_{,p}) + i\omega e_{mn} x_p (\bar{\Phi}_{n,m} \Lambda_{,p} - \Phi_{n,m} \bar{\Lambda}_{,p}) + \\ &+ i\omega m x_p (\bar{\Sigma} \Lambda_{,p} - \Sigma \bar{\Lambda}_{,p}) = -x_p (\frac{c}{\theta_0} \omega^2 \Lambda \bar{\Lambda})_{,p} - [\frac{1}{\theta_0} x_p K_{mn} (\bar{\Lambda}_{,p} \Lambda_{,n} + \Lambda_{,p} \bar{\Lambda}_{,n})]_{,m} + \\ &+ \frac{2}{\theta_0} K_{mn} \Lambda_{,m} \bar{\Lambda}_{,n} + x_p (\frac{1}{\theta_0} K_{mn} \Lambda_{,m} \bar{\Lambda}_{,n})_{,p} \ , \end{aligned} \tag{45}$$

relation that can be rewritten as follows

$$\begin{aligned} &\frac{1}{\theta_0} K_{mn} \Lambda_{,m} \bar{\Lambda}_{,n} - \frac{3c}{\theta_0} \omega^2 \Lambda \bar{\Lambda} + i\omega d_{mn} x_p (\bar{\mathcal{V}}_{n,m} \Lambda_{,p} - \mathcal{V}_{n,m} \bar{\Lambda}_{,p}) + i\omega e_{mn} x_p (\bar{\Phi}_{n,m} \Lambda_{,p} - \Phi_{n,m} \bar{\Lambda}_{,p}) + \\ &+ i\omega m x_p (\bar{\Sigma} \Lambda_{,p} - \Sigma \bar{\Lambda}_{,p}) = -(x_p \frac{c}{\theta_0} \omega^2 \Lambda \bar{\Lambda})_{,p} - [\frac{1}{\theta_0} x_p K_{mn} (\bar{\Lambda}_{,p} \Lambda_{,n} + \Lambda_{,p} \bar{\Lambda}_{,n})]_{,m} + \\ &+ (\frac{x_p}{\theta_0} K_{mn} \Lambda_{,m} \bar{\Lambda}_{,n})_{,p}. \end{aligned} \tag{46}$$

The previous equality (**Error! Reference source not found.**) is integrated over $D(x_3)$ and the boundary conditions ((17)) are used, which leads to

$$\int_{D(x_3)} \frac{1}{\theta_0}(K_{mn}\Lambda_{,m}\bar{\Lambda}_{,n} - 3c\omega\Lambda\bar{\Lambda})dA + \int_{D(x_3)} i\omega d_{mn}x_p(\bar{\mathcal{V}}_{n,m}\Lambda_{,p} - \mathcal{V}_{n,m}\bar{\Lambda}_{,p})dA +$$
$$+ \int_{D(x_3)} i\omega e_{mn}x_p(\bar{\Phi}_{n,m}\Lambda_{,p} - \Phi_{n,m}\bar{\Lambda}_{,p})dA + \int_{D(x_3)} i\omega m x_p(\bar{\Sigma}\Lambda_{,p} - \Sigma\bar{\Lambda}_{,p})dA =$$
$$= -\frac{d}{dx_3}\int_{D(x_3)} [\frac{1}{\theta_0}x_pK_{3n}(\bar{\Lambda}_{,p}\Lambda_{,n} + \Lambda_{,p}\bar{\Lambda}_{,n}) - \frac{x_3}{\theta_0}K_{mn}\Lambda_{,m}\Lambda_{,n} + \frac{x_3}{\theta_0}c\omega^2\Lambda\bar{\Lambda}]dA + \tag{46}$$
$$+ \int_{\partial D(x_3)} \frac{1}{\theta_0}[x_pK_{mn}\Lambda_{,m}\Lambda_{,n}n_p - x_pK_{pn}(\bar{\Lambda}_{,p}\Lambda_{,n} + \Lambda_{,p}\bar{\Lambda}_{,n})n_p]ds.$$

Using the boundary conditions ((17)) in a way similar to the one used to demonstrate the relation, it follows

$$\Lambda_{,3} = 0,\ \ \Lambda_{,\alpha} = n_\alpha\frac{\partial\Lambda}{\partial n},\quad \text{onthecurve}\partial D(x_3). \tag{47}$$

It is observed that the equality ((46)) implies the equality ((33)), which means that the proof of the Theorem 2 is complete. □

4. Main results

The object of the proof for the theorem below is constituted by the conservation laws, which will lead to obtaining „a priori” estimates for a solution of the mixed initial boundary value problem in our context.

Theorem 3. *If $(V_m, \Phi_m, \Sigma, \Lambda)$ is a solution of the boundary value problem consisting of the equation ((16))–((18)), the following two conservation laws take place*

$$\frac{d}{dx_3}\int_{D(x_3)} \omega^2(\rho V_j\bar{V}_j + I_{mn}\Phi_m\bar{\Phi}_n + \rho k\Sigma\bar{\Sigma} + \frac{c}{\theta_0}\Lambda\bar{\Lambda})dA +$$
$$+ \frac{d}{dx_3}\int_{D(x_3)} \frac{1}{\theta_0}(K_{33}\Lambda_{,3}\bar{\Lambda}_{,3} - K_{\alpha\beta}\Lambda_{,\alpha}\bar{\Lambda}_{,\beta})dA + \frac{d}{dx_3}\int_{D(x_3)} [C_{i3m3}\mathcal{V}_{i,3}\bar{\mathcal{V}}_{m,3} +$$
$$+B_{i3m3}(\mathcal{V}_{i,3}\bar{\Phi}_{m,3} + \bar{\mathcal{V}}_{i,3}\Phi_{m,3}) + A_{i3m3}\Phi_{i,3}\bar{\Phi}_{m,3} +$$
$$+B_{i3}(\Sigma\bar{\mathcal{V}}_{i,3} + \bar{\Sigma}\mathcal{V}_{i,3}) + C_{i3}(\Sigma\bar{\Phi}_{i,3} + \bar{\Sigma}\Phi_{i,3}) + A_{i3}(\Sigma_{,i}\bar{\Sigma}_{,3} + \bar{\Sigma}_{,i}\Sigma_{,3})]dA - \tag{48}$$
$$-\frac{d}{dx_3}\int_{D(x_3)} [C_{i\alpha m\beta}\mathcal{V}_{i,\alpha}\bar{\mathcal{V}}_{m,\beta} + B_{i\alpha m\beta}(\mathcal{V}_{i,\alpha}\bar{\Phi}_{m,\beta} + \bar{\mathcal{V}}_{i,\alpha}\Phi_{m,\beta}) + A_{i\alpha m\beta}\Phi_{i,\alpha}\bar{\Phi}_{m,\beta} +$$
$$+B_{i\alpha}(\mathcal{V}_{i,\alpha}\bar{\Sigma} + \bar{\mathcal{V}}_{i,\alpha}\Sigma) + C_{i\alpha}(\Phi_{i,\alpha}\bar{\Sigma} + \bar{\Phi}_{i,\alpha}\Sigma) + A_{i\alpha}(\Sigma_{,i}\bar{\Sigma}_{,\alpha} + \bar{\Sigma}_{,i}\Sigma_{,\alpha})]dA +$$
$$+\frac{d}{dx_3}\int_{D(x_3)} [i\omega d_{i\alpha}(\Lambda\bar{\mathcal{V}}_{i,\alpha} - \bar{\Lambda}\mathcal{V}_{i,\alpha}) + i\omega e_{i\alpha}(\Lambda\bar{\Phi}_{i,\alpha} - \bar{\Lambda}\Phi_{i,\alpha})]dA +$$
$$+\int_{D(x_3)} i\omega m(\Lambda_{,3}\bar{\Sigma} - \bar{\Lambda}_{,3}\Sigma)dA = 0.$$

$$\frac{d}{dx_3}\int_{D(x_3)} \{[C_{3jmn}\bar{\mathcal{V}}_{n,m} + B_{3jmn}\bar{\Phi}_{n,m} + B_{3j}\bar{\Sigma} + id_{3j}\bar{\Lambda}]V_j\}dA -$$
$$-\frac{d}{dx_3}\int_{D(x_3)} \{[C_{3jmn}\mathcal{V}_{n,m} + B_{3jmn}\Phi_{n,m} + B_{3j}\Sigma - id_{3j}\Lambda]\bar{V}_j\}dA +$$
$$+\frac{d}{dx_3}\int_{D(x_3)} \{[B_{3jmn}\bar{\mathcal{V}}_{n,m} + A_{3jmn}\bar{\Phi}_{n,m} + C_{3j}\bar{\Sigma} + i\omega e_{3j}\bar{\Lambda}]\Phi_j\}dA -$$
$$-\frac{d}{dx_3}\int_{D(x_3)} \{[B_{3jmn}\mathcal{V}_{n,m} + A_{3jmn}\Phi_{n,m} + C_{3j}\Sigma - i\omega e_{3j}\Lambda]\bar{\Phi}_j\}dA + \tag{49}$$
$$+\frac{d}{dx_3}\int_{D(x_3)} (A_{3j}\bar{\Sigma}_{,j})\Sigma dA - \frac{d}{dx_3}\int_{D(x_3)} (A_{3j}\Sigma_{,j})\bar{\Sigma}dA =$$
$$= \frac{d}{dx_3}\int_{D(x_3)} \frac{1}{\theta_0}K_{3j}(\bar{\Lambda}\Lambda_{,j} - \Lambda\bar{\Lambda}_{,j})dA + \int_{D(x_3)} [B_{mn}(\bar{\Sigma}\mathcal{V}_{n,m} - \Sigma\bar{\mathcal{V}}_{n,m}) +$$
$$+C_{mn}(\bar{\Sigma}\Phi_{m,n} - \Sigma\bar{\Phi}_{m,n}) - i\omega m(\bar{\Sigma}\Lambda + \Sigma\bar{\Lambda})]dA.$$

Proof. To demonstrate the equality ((48)), the relations $((16))_1$, $((16))_2$ and $((16))_3$ are used, through which the next equality is obtained:

$$
\begin{aligned}
&\{[C_{ijmn}\mathcal{V}_{n,m}+B_{ijmn}\Phi_{n,m}+B_{ij}\Sigma-i\omega d_{ij}\Lambda]_{,i}+\rho\omega^2 V_j\}\bar{V}_{j,3}+\\
&+\{[B_{ijmn}\mathcal{V}_{n,m}+A_{ijmn}\Phi_{n,m}+C_{ij}\Sigma-i\omega e_{ij}\Lambda]_{,i}+\\
&+\varepsilon_{jik}[C_{ikmn}\mathcal{V}_{n,m}+B_{ikmn}\Phi_{n,m}+B_{ik}\Sigma-i\omega d_{ik}\Lambda]+I_{mj}\omega^2\Phi_m\}\bar{\Phi}_{j,3}+\\
&+[(A_{mn}\Sigma_{,n})_{,m}+\rho k\omega^2\Sigma]\bar{\Sigma}_{,3}+[(A_{mn}\bar{\Sigma}_{,n})_{,m}+\rho k\omega^2\bar{\Sigma}]\Sigma_{,3}+\\
&+\{[C_{ijmn}\bar{\mathcal{V}}_{n,m}+B_{ijmn}\bar{\Phi}_{n,m}+B_{ij}\bar{\Sigma}+i\omega d_{ij}\bar{\Lambda}]_{,i}+\rho\omega^2\bar{V}_j\}V_{j,3}+\\
&+\{[B_{ijmn}\bar{\mathcal{V}}_{n,m}+A_{ijmn}\bar{\Phi}_{n,m}+C_{ij}\bar{\Sigma}+i\omega e_{ij}\bar{\Lambda}]_{,i}+\\
&+\varepsilon_{jik}[C_{ikmn}\bar{\mathcal{V}}_{n,m}+B_{ikmn}\bar{\Phi}_{n,m}+B_{ik}\bar{\Sigma}+i\omega d_{ik}\bar{\Lambda}]+I_{mj}\omega^2\Phi_m\}\Phi_{j,3}=0.
\end{aligned}
\tag{50}
$$

The previous relation leads to

$$
\begin{aligned}
&\frac{d}{dx_3}[\rho\omega^2 V_j\bar{V}_j+I_{mn}\omega^2\Phi_m\Phi_n+\rho k\omega^2\Sigma\bar{\Sigma}+C_{i3m3}\mathcal{V}_{i,3}\bar{\mathcal{V}}_{m,3}+\\
&+B_{i3m3}(\mathcal{V}_{i,3}\bar{\Phi}_{m,3}+\bar{\mathcal{V}}_{i,3}\Phi_{m,3})+A_{i3m3}\Phi_{i,3}\bar{\Phi}_{m,3}+B_{i3}(\Sigma\bar{\mathcal{V}}_{i,3}+\bar{\Sigma}\mathcal{V}_{i,3})+\\
&+C_{i3}(\Sigma\bar{\Phi}_{i,3}+\bar{\Sigma}\Phi_{i,3})+A_{i3}(\Sigma_{,i}\bar{\Sigma}_{,3}+\bar{\Sigma}_{,i}\Sigma_{,3})-\\
&-C_{i\alpha m\beta}\mathcal{V}_{i,\alpha}\bar{\mathcal{V}}_{m,\beta}-B_{i\alpha m\beta}(\mathcal{V}_{i,\alpha}\bar{\Phi}_{m,\beta}+\bar{\mathcal{V}}_{i,\alpha}\Phi_{m,\beta})-A_{i\alpha m\beta}\Phi_{i,\alpha}\bar{\Phi}_{m,\beta}-\\
&-B_{i\alpha}(\bar{\mathcal{V}}_{i,\alpha}\Sigma+\mathcal{V}_{i,\alpha}\bar{\Sigma})-C_{i\alpha}(\bar{\Phi}_{i,\alpha}\Sigma+\Phi_{i,\alpha}\bar{\Sigma})-A_{i\alpha}(\Sigma_{,i}\bar{\Sigma}_{,\alpha}+\bar{\Sigma}_{,i}\Sigma_{,\alpha})+\\
&+i\omega d_{i\alpha}(\Lambda\bar{\mathcal{V}}_{i,\alpha}-\bar{\Lambda}\mathcal{V}_{i,\alpha})+i\omega e_{i\alpha}(\Lambda\bar{\Phi}_{i,\alpha}-\bar{\Lambda}\Phi_{i,\alpha})]+\\
&+[C_{i\alpha m3}\mathcal{V}_{m,3}\bar{\mathcal{V}}_{i,\alpha}+B_{i\alpha m3}(\mathcal{V}_{m,3}\bar{\Phi}_{i,\alpha}+\bar{\mathcal{V}}_{m,3}\Phi_{i,\alpha})+A_{i\alpha m3}\Phi_{m,3}\bar{\Phi}_{i,\alpha}+\\
&+B_{i\alpha}(\Sigma\bar{\mathcal{V}}_{i,3}+\bar{\Sigma}\mathcal{V}_{i,3})+C_{i\alpha}(\Sigma\bar{\Phi}_{i,3}+\bar{\Sigma}\Phi_{i,3})+A_{i\alpha}(\Sigma_{,i}\bar{\Sigma}_{,3}+\bar{\Sigma}_{,i}\Sigma_{,3})+\\
&+i\omega d_{i\alpha}(\bar{\Lambda}\mathcal{V}_{i,3}-\Lambda\bar{\mathcal{V}}_{i,3})+i\omega e_{i\alpha}(\bar{\Lambda}\Phi_{i,3}-\Lambda\bar{\Phi}_{i,3})]_{,\alpha}+\\
&+i\omega d_{mn}(\bar{\Lambda}_{,3}\mathcal{V}_{m,n}-\Lambda_{,3}\bar{\mathcal{V}}_{m,n})+i\omega e_{mn}(\bar{\Lambda}_{,3}\Phi_{m,n}-\Lambda_{,3}\bar{\Phi}_{m,n})=0.
\end{aligned}
\tag{51}
$$

Integrating the above equality over $D(x_3)$ and using the boundary conditions ((17)), we get

$$
\begin{aligned}
&\frac{d}{dx_3}\int_{D(x_3)}[\rho\omega^2 V_j\bar{V}_j+I_{mn}\omega^2\Phi_m\bar{\Phi}_n+\rho k\omega^2\Sigma\bar{\Sigma}+C_{i3m3}\mathcal{V}_{i,3}\bar{\mathcal{V}}_{m,3}+\\
&+B_{i3m3}(\mathcal{V}_{i,3}\bar{\Phi}_{m,3}+\bar{\mathcal{V}}_{i,3}\Phi_{m,3})+A_{i3m3}\Phi_{i,3}\bar{\Phi}_{m,3}+B_{i3}(\Sigma\bar{\mathcal{V}}_{i,3}+\bar{\Sigma}\mathcal{V}_{i,3})+\\
&+C_{i3}(\Sigma\bar{\Phi}_{i,3}+\bar{\Sigma}\Phi_{i,3})+A_{i3}(\Sigma_{,i}\bar{\Sigma}_{,3}+\bar{\Sigma}_{,i}\Sigma_{,3})-\\
&-C_{i\alpha m\beta}\mathcal{V}_{i,\alpha}\bar{\mathcal{V}}_{m,\beta}-B_{i\alpha m\beta}(\mathcal{V}_{i,\alpha}\bar{\Phi}_{m,\beta}+\bar{\mathcal{V}}_{i,\alpha}\Phi_{m,\beta})-A_{i\alpha m\beta}\Phi_{i,\alpha}\bar{\Phi}_{m,\beta}-\\
&-B_{i\alpha}(\bar{\mathcal{V}}_{i,\alpha}\Sigma+\mathcal{V}_{i,\alpha}\bar{\Sigma})-C_{i\alpha}(\bar{\Phi}_{i,\alpha}\Sigma+\Phi_{i,\alpha}\bar{\Sigma})-A_{i\alpha}(\Sigma_{,i}\bar{\Sigma}_{,\alpha}+\bar{\Sigma}_{,i}\Sigma_{,\alpha})+\\
&+i\omega d_{i\alpha}(\Lambda\bar{\mathcal{V}}_{i,\alpha}-\bar{\Lambda}\mathcal{V}_{i,\alpha})+i\omega e_{i\alpha}(\Lambda\bar{\Phi}_{i,\alpha}-\bar{\Lambda}\Phi_{i,\alpha})]dA+\\
&+\int_{D(x_3)}[i\omega d_{mn}(\bar{\Lambda}_{,3}\mathcal{V}_{m,n}-\Lambda_{,3}\bar{\mathcal{V}}_{m,n})+i\omega e_{mn}(\bar{\Lambda}_{,3}\Phi_{m,n}-\Lambda_{,3}\bar{\Phi}_{m,n})]dA=0.
\end{aligned}
\tag{52}
$$

By using the equation ((16)) $_3$, the following equality remains fulfilled:

$$
\begin{aligned}
&\bar{\Lambda}_{,3}[(\frac{1}{\theta_0}K_{mn}\Lambda_{,n})_{,m}-i\omega d_{mn}\mathcal{V}_{n,m}-i\omega e_{mn}\Phi_{n,m}-i\omega m\Sigma+\frac{c}{\theta_0}\omega^2\Lambda]+\\
&+\Lambda_{,3}[(\frac{1}{\theta_0}K_{mn}\bar{\Lambda}_{,n})_{,m}+i\omega d_{mn}\bar{\mathcal{V}}_{n,m}+i\omega e_{mn}\bar{\Phi}_{n,m}+i\omega m\bar{\Sigma}+\frac{c}{\theta_0}\omega^2\bar{\Lambda}]=0.
\end{aligned}
\tag{53}
$$

The previous relation ((53)) can also be written in the form below

$$
\begin{aligned}
&\frac{d}{dx_3}(\frac{c}{\theta_0}\omega^2\Lambda\bar{\Lambda}+\frac{1}{\theta_0}K_{33}\Lambda_{,3}\bar{\Lambda}_{,3}-\frac{1}{\theta_0}K_{\alpha\beta}\Lambda_{,\alpha}\bar{\Lambda}_{,\beta})+(\frac{2}{\theta}K_{\alpha 3}\Lambda_{,3}\bar{\Lambda}_{,3})_{,\alpha}+\\
&+i\omega d_{mn}(\Lambda_{,3}\bar{\mathcal{V}}_{m,n}-\bar{\Lambda}_{,3}\mathcal{V}_{m,n})+i\omega e_{mn}(\Lambda_{,3}\bar{\Phi}_{m,n}-\bar{\Lambda}_{,3}\Phi_{m,n})+i\omega m(\Lambda_{,3}\bar{\Sigma}-\bar{\Lambda}_{,3}\Sigma)=0.
\end{aligned}
\tag{54}
$$

If the relation ((54)) is integrated over $D(x_3)$ and the conditions ((17)) are used, the next equality is obtained:

$$
\begin{aligned}
&\frac{d}{dx_3}\int_{D(x_3)}(\frac{c}{\theta_0}\omega^2\Lambda\bar{\Lambda}+\frac{1}{\theta_0}K_{33}\Lambda_{,3}\bar{\Lambda}_{,3}-\frac{1}{\theta_0}K_{\alpha\beta}\Lambda_{,\alpha}\bar{\Lambda}_{,\beta})dA+\\
&+\int_{D(x_3)}[i\omega d_{mn}(\Lambda_{,3}\bar{\mathcal{V}}_{m,n}-\bar{\Lambda}_{,3}\mathcal{V}_{m,n})+i\omega e_{mn}(\Lambda_{,3}\bar{\Phi}_{m,n}-\bar{\Lambda}_{,3}\Phi_{m,n})+i\omega m(\Lambda_{,3}\bar{\Sigma}-\bar{\Lambda}_{,3}\Sigma)]dA=0.
\end{aligned}
$$

By means of the relations ((52)) and ((55)), the equality ((48)) is reached, and through the equalities ((21) and ((23)) the conservation law (50) is determined, this completes the proof of Theorem 3, which is now concluded.

The result presented in the following represents the first estimate that characterizes the solution spatial behaviour. □

Theorem 4. *If* $(V_m, \Phi_m, \Sigma, \Lambda)$ *is a solution of the boundary value problem represented by the equations ((16))–((18)), the following equality is satisfied:*

$$2\int_{D(x_3)} [C_{ijmn}\mathcal{V}_{j,i}\bar{\mathcal{V}}_{n,m} + B_{ijmn}(\mathcal{V}_{j,i}\bar{\Phi}_{n,m} + \bar{\mathcal{V}}_{j,i}\Phi_{n,m}) + A_{ijmn}\Phi_{n,m}\bar{\Phi}_{j,i} + A_{mn}\Sigma_{,m}\bar{\Sigma}_{,n} -$$
$$-\omega^2(\rho V_m\bar{V}_m + I_{mn}\Phi_m\bar{\Phi}_n + \rho k\Sigma\bar{\Sigma} + \frac{c}{\theta_0}\Lambda\bar{\Lambda}) + \frac{1}{\theta_0}K_{mn}\Lambda_{,m}\bar{\Lambda}_{,n} +$$
$$+i\omega d_{mn}(\mathcal{V}_{n,m}\bar{\Lambda} - \bar{\mathcal{V}}_{n,m}\Lambda) + i\omega e_{mn}(\bar{\Lambda}\Phi_{m,n} - \Lambda\bar{\Phi}_{m,n})]dA +$$
$$+\int_{D(x_3)} [B_{mn}(\Sigma\bar{\mathcal{V}}_{n,m} + \bar{\Sigma}\mathcal{V}_{n,m}) + C_{mn}(\Sigma\bar{\Phi}_{m,n} + \bar{\Sigma}\Phi_{m,n}) + i\omega m(\Sigma\bar{\Lambda} - \bar{\Sigma}\Lambda)]dA =$$
$$= \frac{d}{dx_3}\int_{D(x_3)} \{[C_{3jmn}\mathcal{V}_{n,m} + B_{3jmn}\Phi_{n,m} + B_{3j}\Sigma - i\omega d_{3j}\Lambda]\bar{V}_j\}dA + \tag{56}$$
$$+\frac{d}{dx_3}\int_{D(x_3)} \{[C_{3jmn}\bar{\mathcal{V}}_{n,m} + B_{3jmn}\bar{\Phi}_{n,m} + B_{3j}\bar{\Sigma} + i\omega d_{3j}\bar{\Lambda}]V_j\}dA +$$
$$+\frac{d}{dx_3}\int_{D(x_3)} \{[B_{3jmn}\mathcal{V}_{n,m} + A_{3jmn}\Phi_{n,m} + C_{3j}\Sigma - i\omega e_{3j}\Lambda]\bar{\Phi}_j\}dA +$$
$$+\frac{d}{dx_3}\int_{D(x_3)} \{[B_{3jmn}\bar{\mathcal{V}}_{n,m} + A_{3jmn}\bar{\Phi}_{n,m} + C_{3j}\bar{\Sigma} + i\omega e_{3j}\bar{\Lambda}]\Phi_j\}dA +$$
$$+\frac{d}{dx_3}\int_{D(x_3)} [(A_{3j}\Sigma_{,j})\bar{\Sigma} + (A_{3j}\bar{\Sigma}_{,j})\Sigma]dA + \frac{d}{dx_3}\int_{D(x_3)} \frac{1}{\theta_0}K_{3j}(\bar{\Lambda}\Lambda_{,j} + \Lambda\bar{\Lambda}_{,j})dA.$$

Proof. The use of the equations (20) and ((22)) leads to the immediate obtaining of the relation (57). □

In what follows, a theorem that establishes another estimate will be presented.

Theorem 5. *If* $(V_m, \Phi_m, \Sigma, \Lambda)$ *is a solution of the boundary value problem consisting of the equations ((16))–((18)), then the next equality is fulfilled:*

$$\int_{D(x_3)} [C_{ijmn}\mathcal{V}_{j,i}\bar{\mathcal{V}}_{n,m} + B_{ijmn}(\mathcal{V}_{j,i}\bar{\Phi}_{n,m} + \bar{\mathcal{V}}_{j,i}\Phi_{n,m}) + A_{ijmn}\Phi_{i,j}\bar{\Phi}_{n,m} + A_{mn}\Sigma_{,m}\bar{\Sigma}_{,n} +$$
$$+\frac{1}{\theta_0}K_{mn}\Lambda_{,m}\bar{\Lambda}_{,n} + \omega^2(\rho V_m\bar{V}_m + I_{mn}\Phi_m\bar{\Phi}_n + \rho k\Sigma\bar{\Sigma} + \frac{c}{\theta_0}\Lambda\bar{\Lambda})]dA -$$
$$-\int_{\partial D(x_3)} x_p n_p (C_{i\alpha m\beta}n_\alpha n_\beta \frac{\partial \mathcal{V}_i}{\partial n}\frac{\partial \bar{\mathcal{V}}_m}{\partial n} + 2B_{i\alpha m\beta}n_\alpha n_\beta \frac{\partial \mathcal{V}_i}{\partial n}\frac{\partial \bar{\Phi}_m}{\partial n} +$$
$$+A_{i\alpha m\beta}n_\alpha n_\beta \frac{\partial \Phi_i}{\partial n}\frac{\partial \bar{\Phi}_m}{\partial n} + A_{\alpha\beta}n_\alpha n_\beta \frac{\partial \Sigma}{\partial n}\frac{\partial \bar{\Sigma}}{\partial n})ds - \int_{\partial D(x_3)} x_p n_p K_{\alpha\beta}n_\alpha n_\beta \frac{\partial \Lambda}{\partial n}\frac{\partial \bar{\Lambda}}{\partial n}ds =$$
$$= \frac{d}{dx_3}\int_{D(x_3)} [(C_{3jmn}\mathcal{V}_{n,m} + B_{3jmn}\Phi_{n,m} + B_{3j}\Sigma - i\omega d_{3j}\Lambda)(\bar{V}_j + x_p\bar{V}_{j,p}) +$$
$$+(C_{3jmn}\bar{\mathcal{V}}_{n,m} + B_{3jmn}\bar{\Phi}_{n,m} + B_{3j}\bar{\Sigma} + i\omega d_{3j}\bar{\Lambda})(V_j + x_p V_{j,p})]dA +$$
$$+\frac{d}{dx_3}\int_{D(x_3)} [(B_{3jmn}\mathcal{V}_{n,m} + A_{3jmn}\Phi_{n,m} + C_{3j}\Sigma - i\omega e_{3j}\Lambda)(\bar{\Phi}_j + x_p\bar{\Phi}_{j,p}) +$$
$$+(B_{3jmn}\bar{\mathcal{V}}_{n,m} + A_{3jmn}\bar{\Phi}_{n,m} + C_{3j}\bar{\Sigma} + i\omega e_{3j}\bar{\Lambda})(\Phi_j + x_p\Phi_{j,p})]dA + \tag{57}$$
$$+\frac{d}{dx_3}\int_{D(x_3)} [(A_{3j}\Sigma_{,j})(\bar{\Sigma} + x_p\bar{\Sigma}_{,p}) + (A_{3j}\bar{\Sigma}_{,j})(\Sigma + x_p\Sigma_{,p})]dA +$$
$$+\frac{d}{dx_3}\int_{D(x_3)} \frac{1}{\theta_0}[K_{33}(\Lambda_{,3}\bar{\Lambda} + \bar{\Lambda}_{,3}\Lambda) + K_{3\alpha}(\Lambda_{,\alpha}\bar{\Lambda} + \bar{\Lambda}_{,\alpha}\Lambda)]dA +$$
$$+\frac{d}{dx_3}\int_{D(x_3)} \frac{x_\alpha}{\theta_0}[K_{3\beta}(\Lambda_{,\beta}\bar{\Lambda}_{,\alpha} + \bar{\Lambda}_{,\beta}\Lambda_{,\alpha}) + K_{33}(\Lambda_{,3}\bar{\Lambda}_{,\alpha} + \bar{\Lambda}_{,3}\Lambda_{,\alpha})]dA -$$

$$-\frac{d}{dx_3}\int_{D(x_3)}\{x_3[C_{i3m3}\mathcal{V}_{i,3}\bar{\mathcal{V}}_{m,3}+B_{i3m3}(\mathcal{V}_{i,3}\bar{\Phi}_{m,3}+\bar{\mathcal{V}}_{i,3}\Phi_{m,3})+A_{i3m3}\Phi_{i,3}\bar{\Phi}_{m,3}+A_{33}\Sigma_{,3}\bar{\Sigma}_{,3}]+$$
$$+x_3[C_{i\alpha m\beta}\mathcal{V}_{i,\alpha}\bar{\mathcal{V}}_{m,\beta}+B_{i\alpha m\beta}(\mathcal{V}_{i,\alpha}\bar{\Phi}_{m,\beta}+\bar{\mathcal{V}}_{i,\alpha}\Phi_{m,\beta})+A_{i\alpha m\beta}\Phi_{i,\alpha}\bar{\Phi}_{m,\beta}+A_{\alpha\beta}\Sigma_{,\alpha}\bar{\Sigma}_{,\beta}]$$
$$+x_3 i\omega[D_{i\alpha}(T\bar{\mathcal{V}}_{i,\alpha}-\bar{T}\mathcal{V}_{i,\alpha})+E_{i\alpha}(T\bar{\Phi}_{i,\alpha}-\bar{T}\Phi_{i,\alpha})+(\Sigma\bar{\Lambda}-\bar{\Sigma}\Lambda)]\}dA$$
$$+\frac{d}{dx_3}\int_{D(x_3)}[\frac{x_3}{\theta_0}(K_{33}\Lambda_{,3}\bar{\Lambda}_{,3}-K_{\alpha\beta}\Lambda_{,\alpha}\bar{\Lambda}_{,\beta})+x_3\omega^2(\rho V_m\bar{V}_m+I_{mn}\Phi_m\bar{\Phi}_n+\rho k\Sigma\bar{\Sigma}+\frac{c}{\theta_0}\Lambda\bar{\Lambda})]dA.$$

Proof. The equality (58) is obtained by using the results represented by equalities (32) and (33) of the Theorem 2, together with the equation (57) of the Theorem 4.

The relation (58) will constitute the foundation for deducing the conclusions regarding the spatial behaviour of the amplitude $(V_m,\Phi_m,\Sigma,\Lambda)$, and for accuracy, it will be assumed that the micropolar thermoelasticity tensors satisfy the usual hypotheses of continuum mechanics, namely, they satisfy the strong ellipticity conditions:

$$\begin{aligned}&C_{ijmn}x_ix_my_iy_n>0,\\&B_{ijmn}x_ix_my_iy_n>0,\\&A_{ijmn}x_ix_my_iy_n>0,\end{aligned}\tag{58}$$

for all non-null vectors (x_1,x_2,x_3), (y_1,y_2,y_3), and the specific heat c, the coefficients A_{mn} and the components of the conductivity tensor K_{mn} fulfill the conditions below

$$\begin{aligned}&c>0,\\&A_{mn}x_mx_n>0,\\&K_{mn}x_mx_n>0,\end{aligned}\tag{59}$$

for all non-null vectors (x_1,x_2,x_3). Using the relation ((58)), the following deduction are evident

$$\begin{aligned}&C_{i3m3}x_ix_m>0,\\&B_{i3m3}x_ix_m>0,\\&A_{i3m3}x_ix_m>0,\end{aligned}\tag{60}$$

for all non-null vectors (x_1,x_2,x_3). The curve ∂D being assumed to be regular, it follows that there exists $s_0>0$ such that $0<s_0\le x_pn_p$.

The following inequalities will be satisfied:

$$\begin{aligned}&0\le\int_{\partial D(x_3)}x_pn_p(C_{i\alpha m\beta}n_\alpha n_\beta\frac{\partial\mathcal{V}_i}{\partial n}\frac{\partial\bar{\mathcal{V}}_m}{\partial n}+2B_{i\alpha m\beta}n_\alpha n_\beta\frac{\partial\mathcal{V}_i}{\partial n}\frac{\partial\bar{\Phi}_m}{\partial n}+A_{i\alpha m\beta}n_\alpha n_\beta\frac{\partial\Phi_i}{\partial n}\frac{\partial\bar{\Phi}_m}{\partial n}+\\&+A_{\alpha\beta}n_\alpha n_\beta\frac{\partial\Sigma}{\partial n}\frac{\partial\bar{\Sigma}}{\partial n})ds\le MC\int_{\partial D(x_3)}(\frac{\partial\mathcal{V}_m}{\partial n}\frac{\partial\bar{\mathcal{V}}_m}{\partial n}+\frac{\partial\Phi_m}{\partial n}\frac{\partial\bar{\Phi}_m}{\partial n}+\frac{\partial\Sigma}{\partial n}\frac{\partial\bar{\Sigma}}{\partial n})ds,\end{aligned}\tag{61}$$

where

$$\begin{aligned}&M=\sup_{(x_1,x_2)\in\partial D}\sqrt{(x_1^2+x_2^2)},\\&C=(C_{i\alpha m\beta}C_{i\alpha m\beta}+2B_{i\alpha m\beta}B_{i\alpha m\beta}+A_{i\alpha m\beta}A_{i\alpha m\beta}+A_{\alpha\beta}A_{\alpha\beta})^{\frac{1}{2}}.\end{aligned}\tag{62}$$

At the same time, the inequalities related to the conductivity tensor are fulfilled, namely:

$$0\le\int_{\partial D(x_3)}\frac{1}{\theta_0}x_pn_pK_{\alpha\beta}n_\alpha n_\beta\frac{\partial\Lambda}{\partial n}\frac{\partial\bar{\Lambda}}{\partial n}ds\le\frac{MK}{\theta_0}\int_{\partial D(x_3)}\frac{\partial\Lambda}{\partial n}\frac{\partial\bar{\Lambda}}{\partial n}ds,\tag{63}$$

where M is defined by the relation ((62)) $_1$, and

$$K=(K_{\alpha\beta}K_{\alpha\beta})^{\frac{1}{2}}.\tag{64}$$

Next, enter the quantities m_0,m_1,ω_0^* and ω_1^* as follows

$$m_0 = \max_{x_3\in[0,L]} \frac{\int_{\partial D(x_3)} \left(\frac{\partial \mathcal{V}_m}{\partial n}\frac{\partial \bar{\mathcal{V}}_m}{\partial n} + \frac{\partial \Phi_m}{\partial n}\frac{\partial \bar{\Phi}_m}{\partial n} + \frac{\partial \Sigma}{\partial n}\frac{\partial \bar{\Sigma}}{\partial n}\right) ds}{\int_{D(x_3)} (V_m \bar{V}_m + \Phi_m \bar{\Phi}_m + \Sigma\bar{\Sigma}) dA}, \tag{65}$$

$$m_1 = \max_{x_3\in[0,L]} \frac{\int_{\partial D(x_3)} \frac{\partial \Lambda}{\partial n}\frac{\partial \bar{\Lambda}}{\partial n} ds}{\int_{D(x_3)} \Lambda\bar{\Lambda} dA}, \tag{66}$$

$$\omega^* = \frac{1}{\rho} MCm_0, \tag{67}$$

$$\omega_1^* = \frac{1}{c} MKm_1. \tag{68}$$

It is assumed that

$$\omega > \omega^* = max\{\omega_0^*, \omega_1^*\}, \tag{69}$$

$$m \le m_0^*, \quad m_1 \le m_1^*, \tag{70}$$

where

$$m_0^* = \max \frac{\int_{\partial D(x_3)} \left(\frac{\partial \mathcal{V}_m}{\partial n}\frac{\partial \bar{\mathcal{V}}_m}{\partial n} + \frac{\partial \Phi_m}{\partial n}\frac{\partial \bar{\Phi}_m}{\partial n} + \frac{\partial \Sigma}{\partial n}\frac{\partial \bar{\Sigma}}{\partial n}\right) ds}{\int_{D(x_3)} (V_m \bar{V}_m + \Phi_m \bar{\Phi}_m + \Sigma\bar{\Sigma}) dA}, \tag{71}$$

$$m_1 = \max_{\Lambda\in H_0^1(D)} \frac{\int_{\partial D(x_3)} \frac{\partial \Lambda}{\partial n}\frac{\partial \bar{\Lambda}}{\partial n} ds}{\int_{D(x_3)} \Lambda\bar{\Lambda} dA}. \tag{72}$$

For expression of m_0^*, represented by the relation (71), the maximum is determined for $V_m \in H_0^1(D), \Phi_m \in H_0^1(D)$ and $\Sigma \in H_0^1(D)$, $H_0^1(D)$ being the usual Sobolev space, thus determining a critical value for the vibration frequency, presented in the following

$$\omega^* = \max\{\frac{1}{\rho} MCm_0^*, \frac{1}{c} MKm_1^*\}. \tag{73}$$

To estimate the spatial behaviour of the amplitude $(V_m, \Phi_m, \Sigma, \Lambda)$, the relation (58),(61),(63) and (69) are used, obtaining the next inequality:

$$\begin{aligned}
&\frac{d}{dx_3}\int_{D(x_3)} [(C_{3jmn}\mathcal{V}_{n,m} + B_{3jmn}\Phi_{n,m} + B_{3j}\Sigma - i\omega d_{3j}\Lambda)(\bar{V}_j + x_p\bar{V}_{j,p}) + \\
&+(C_{3jmn}\bar{\mathcal{V}}_{n,m} + B_{3jmn}\bar{\Phi}_{n,m} + B_{3j}\bar{\Sigma} + i\omega d_{3j}\bar{\Lambda})(V_j + x_pV_{j,p})]dA + \\
&+\frac{d}{dx_3}\int_{D(x_3)} [(B_{3jmn}\mathcal{V}_{n,m} + A_{3jmn}\Phi_{n,m} + C_{3j}\Sigma - i\omega e_{3j}\Lambda)(\bar{\Phi}_j + x_p\bar{\Phi}_{j,p}) + \\
&+(B_{3jmn}\bar{\mathcal{V}}_{n,m} + A_{3jmn}\bar{\Phi}_{n,m} + C_{3j}\bar{\Sigma} + i\omega e_{3j}\bar{\Lambda})(\Phi_j + x_p\Phi_{j,p})]dA + \\
&+\frac{d}{dx_3}\int_{D(x_3)} [(A_{3j}\Sigma_{,j})(\bar{\Sigma} + x_p\bar{\Sigma}_{,p}) + (A_{3j}\bar{\Sigma}_{,j})(\Sigma + x_p\Sigma_{,p})]dA + \\
&+\frac{d}{dx_3}\int_{D(x_3)} \frac{1}{\theta_0}[K_{33}(\Lambda_{,3}\bar{\Lambda} + \bar{\Lambda}_{,3}\Lambda) + K_{3\alpha}(\Lambda_{,\alpha}\bar{\Lambda} + \bar{\Lambda}_{,\alpha}\Lambda)]dA + \\
&+\frac{d}{dx_3}\int_{D(x_3)} \frac{x_\alpha}{\theta_0}[K_{3\beta}(\Lambda_{,\beta}\bar{\Lambda}_{,\alpha} + \bar{\Lambda}_{,\beta}\Lambda_{,\alpha}) + K_{33}(\Lambda_{,3}\bar{\Lambda}_{,\alpha} + \bar{\Lambda}_{,3}\Lambda_{,\alpha})]dA - \\
&-\frac{d}{dx_3}\int_{D(x_3)} \{x_3[C_{i3m3}\mathcal{V}_{i,3}\bar{\mathcal{V}}_{m,3} + B_{i3m3}(\mathcal{V}_{i,3}\bar{\Phi}_{m,3} + \bar{\mathcal{V}}_{i,3}\Phi_{m,3}) + A_{i3m3}\Phi_{i,3}\bar{\Phi}_{m,3} + A_{33}\Sigma_{,3}\bar{\Sigma}_{,3}] + \\
&+x_3[C_{i\alpha m\beta}\mathcal{V}_{i,\alpha}\bar{\mathcal{V}}_{m,\beta} + B_{i\alpha m\beta}(\mathcal{V}_{i,\alpha}\bar{\Phi}_{m,\beta} + \bar{\mathcal{V}}_{i,\alpha}\Phi_{m,\beta}) + A_{i\alpha m\beta}\Phi_{i,\alpha}\bar{\Phi}_{m,\beta} + A_{\alpha\beta}\Sigma_{,\alpha}\bar{\Sigma}_{,\beta}] + \\
&+x_3 i\omega[D_{i\alpha}(T\bar{\mathcal{V}}_{i,\alpha} - \bar{T}\mathcal{V}_{i,\alpha}) + E_{i\alpha}(T\bar{\Phi}_{i,\alpha} - \bar{T}\Phi_{i,\alpha}) + (\Sigma\bar{\Lambda} - \bar{\Sigma}\Lambda)]\}dA + \\
&+\frac{d}{dx_3}\int_{D(x_3)} [\frac{x_3}{\theta_0}(K_{33}\Lambda_{,3}\bar{\Lambda}_{,3} - K_{\alpha\beta}\Lambda_{,\alpha}\bar{\Lambda}_{,\beta}) + x_3\omega^2(\rho V_m\bar{V}_m + I_{mn}\Phi_m\bar{\Phi}_n + \rho k\Sigma\bar{\Sigma} + \frac{c}{\theta_0}\Lambda\bar{\Lambda})]dA \ge \\
&\ge \int_{D(x_3)} [C_{ijmn}\mathcal{V}_{j,i}\bar{\mathcal{V}}_{n,m} + B_{ijmn}(\mathcal{V}_{j,i}\bar{\Phi}_{n,m} + \bar{\mathcal{V}}_{j,i}\Phi_{n,m}) + A_{ijmn}\Phi_{i,j}\bar{\Phi}_{n,m} + A_{mn}\Sigma_{,m}\bar{\Sigma}_{,n} +
\end{aligned} \tag{74}$$

$$+\frac{1}{\theta_0}K_{mn}\Lambda_{,m}\bar{\Lambda}_{,n}+\omega^2(\rho V_m\bar{V}_m+I_{mn}\Phi_m\bar{\Phi}_n+\rho k\Sigma\bar{\Sigma}+\frac{c}{\theta_0}\Lambda\bar{\Lambda})]dA.$$

The preceding relation (75) represents an estimate of the spatial behaviour of the amplitude $(V_m, \Phi_m, \Sigma, \Lambda)$, which leads to the conclusion that the proof of Theorem 5 is complete. □

5. Conclusions

The study of the spatial evolution of the harmonic in time vibrations, associated with porous micropolar media, is carried out, throughout this article, in a special way from the classic Saint-Venant type estimates, being based only on the conditions of strong ellipticity of the thermoelastic coefficients. The first two theorems lead to the determination of some preliminary identities, which will constitute the basis for obtaining the main results, consisting of estimates of the harmonic vibrations amplitude, including those derived from the influence of the distance from the disturbed base, taking into account a critical value of the vibration frequency.

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