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Metric and partition dimension of flower and pencil graphs

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ABSTRACT

This paper is about metric and partition dimension of a flower and a pencil graph. A *metric dimension* of G, denoted by $dim(G)$, is the minimum cardinality of any resolving set of G. A *partition dimension* of G, denoted by $pd(G)$, is the minimum number of sets in any resolving k-ordered partition for G. Here we give the exact value of the metric dimension of a flower graph $f_{m \times n}$ for $m \in \{3,4\}$ and a pencil graph Pc_m for any integer $m \geq 2$. We also give the partition dimension of $f_{m \times n}$ for $m \in \{3,4,5\}$ and Pc_m for any integer $m \geq 2$.

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1 Introduction

Throughout here, all graphs are simple, undirected, and connected. Let $V(G)$ be the vertex-set of a graph G and $u, v \in V(G)$. The distance from a vertex u to v is minimum

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length of the paths from u to v. Two problems in a graph theory which is considering the distance are metric and partition dimension. Independently, Harary and Melter [14] and Slater [23] described the concept of metric dimension in a graph. Let $W = \{w_1, w_2, \ldots, w_k\}$ be an ordered set of vertices of a graph G. For $x \in V(G)$, the representation $r(x|W)$ of x with respect to W is ordered k-tuple $(d(x, w_1), d(x, w_2), \ldots, d(x, w_k))$. A set W is a resolving set of a graph G if for every two distinct vertices $x, y \in V(G)$, having distinct representation, *i.e.* $r(x|W) \neq r(y|W)$. Clearly that the resolving set of a graph G is not unique. The minimum cardinality of any resolving set of G is a *basis* for G. A basis for G is called the *metric dimension* of a graph G, and denoted by $dim(G)$. Chartrand et al. [7] characterized graph having metric dimension one.

Theorem 1.1. [7] *Suppose G is a connected graph on n vertices.*

- *The only graph having metric dimension* 1 *is a path P_n.*
- *The only graph having metric dimension $n - 1$ is a complete graph K_n.* □

Imran et al. [15] show that the flower graph $f_{m\times 3}$ has bounded metric dimension, while for each $n \geq 4$, the metric dimension of $f_{m\times 3}$ is unbounded as m tends to infinity. Dudenko and Oliynyk [8, 9] characterized all unicyclic graphs having metric dimension 2 with vertices of degree 4. Next, Akhter and Farooq [1] gave the metric dimension of fullerene graphs. Bailey et al. [4] consider the Johnson graphs $J(n, k)$ and Kneser graphs $K(n, k)$, and obtain various constructions of resolving sets for these graphs. Girisha et al. [12] described the metric dimension of sunflower graphs and flower snarks.

Chartrand et al. [6] initiated another problem of dimension of a connected graph, called partition dimension. Let $\Lambda = \{S_1, S_2, \ldots, S_k\}$ be an ordered k-partition of $V(G)$ and S_i be the partition class of $V(G)$ with respect to Λ for each $i \in [1, k]$. A representation $r(x|\Lambda)$ for $x \in V(G)$ with respect to Λ is defined by $r(x|\Lambda) = (d(x, S_1), d(x, S_2), \ldots, d(x, S_k))$, where $d(x, S_i) = \min\{d(x, s)|s \in S_i\}$ for each $i \in [1, k]$. A partition Λ satisfying for every two vertices $x, z \in V(G)$, if $r(x|\Lambda) = r(z|\Lambda)$, then $x = z$, is called a *resolving partition.* The number of elements of a minimum resolving partition of a graph G is called the *partition dimension* of a graph G and denoted by $pd(G)$. They proved that the bounded above of a partition dimension of a graph G is one more than its metric dimension, by the following theorem.

Theorem 1.2. [6] *Suppose that G is a connected graph of order greater than* 1. *Then,* $pd(G) \leq dim(G) + 1$. □

This bound is attainable if G is a path P_n, cycle C_n, complete graph K_n on n vertices or star $K_{1,n}$ on $n + 1$ vertices. A path on n vertices, denoted by P_n, is a connected graph having no cycle with two pendant vertices. A cycle on n vertices, C_n, is a regular connected graph of degree two. A regular connected graph on n vertices of degree $n - 1$ is called a complete graph, K_n. A star $K_{1,n}$ on $n + 1$ vertices is a connected graph with n pendant vertices and a vertex of degree n. A pendant vertex of a graph G is a vertex of degree one. They also characterized graph having partition dimension 2 and n, respectively, by the result below.

Theorem 1.3. [6] *Let $n \geq 2$ be an integer. Suppose G is a connected graph on n vertices.*

- *The only graph having partition dimension 2 is a path P_n.*
- *The only graph having partition dimension n is a complete graph K_n.* □

Next lemma is property on how to determine a resolving partition of a graph.

Lemma 1.4. [6] *Let G be a connected graph, Λ be a resolving partition of $V(G)$ and x and y be two distinct vertices of $V(G)$. If $d(x, v) = d(y, v)$ for all $v \in V(G) - \{x, y\}$, then x and y belong to distinct elements of the resolving partition Λ.* □

Chartrand et al. [6] gave characterization of graphs of order n having partition dimension $n-1$. Tomescu et al. [24] determined the partition dimension of wheel W_n with n spokes. Next, some researchers discussed the partition dimension of trees, see [3, 11, 17]. The partition dimension of unicyclic graphs was studied in [8, 9, 18]. Furthermore, Amrullah et al. [2] discussed the partition dimension of subdivision of a complete graph, $S(K_n)$. Another results about partition dimension of graphs can be seen in [10, 13, 19, 26]. Some researchers studied a relation between metric and partition dimension of a graph, see [5, 20, 21, 25].

Here, we focus on the metric and partition dimension of a graph. We give the exact value of the metric and partition dimension of two classes of graphs, namely flower and pencil graphs. Following Mphako-Banda [16], a *flower* graph, denoted by $f_{m\times n}$, is a graph formed by a cycle C_m and m copies of cycle C_n around the cycle C_m so that each cycle C_n uniquely intersects with the cycle C_m on a single edge. It is clear that a flower $f_{m\times n}$ has $m(n-1)$ vertices and mn edges. Following Simamora and Salman [22], a *pencil* graph on $2m+2$ vertices, denoted by Pc_m is a regular graph of degree 3 having the set of vertices $V(Pc_m) = \{u_0, u_1, \ldots, u_m\} \cup \{v_0, v_1, \ldots, v_m\}$ and the set of edges $E(Pc_m) = \{u_iu_{i+1}, v_iv_{i+1} | i \in [1, m-1]\} \cup \{u_iv_i | i \in [1, m]\} \cup \{u_0u_1, u_0v_1, v_0u_m, v_0v_m\}$. So, a pencil graph Pc_m has $2m+2$ vertices and $3m+3$ edges. Indeed, Imran et al. [15] have given the exact value of the metric dimension of the flower graph $f_{m\times n}$ in theorem below.

Theorem 1.5. [15] *Let m and n be positive integers. Then*

$$dim(f_{m\times n}) = \begin{cases} 2 & \text{for } m \text{ even and } n = 3, \\ 3 & \text{for } m \text{ odd and } n = 3, \\ \lceil \frac{m}{2} \rceil & \text{for } m \geq 5 \text{ and } n \geq 4. \end{cases} \quad \square$$

However, according to Theorem 1.5, there is still open problem for $dim(f_{m\times n})$ for $m \in \{3, 4\}$. In this paper, we will answer these, so that the problem is closed. Beside that, we give the exact value of the partition dimension of flower graphs $f_{m\times 3}$ for all $m \geq 3$ and $f_{m\times n}$ for any integer $n \geq 4$ and $m \in \{3, 4, 5\}$. The metric and partition dimension of a pencil graph Pc_m is also given for any integer $m \geq 2$.

2 Main Results

In this section, we discuss the metric dimension of the flower and pencil graphs. We also talk about the partition dimension of them. We give the exact value of the metric and partition dimension of them.

2.1 Flower graph

For $m \geq 3$ and $n \geq 3$, suppose $V(f_{m\times n}) = \{u^j, v_i^j \mid j \in [1,m], i \in [1, n-2]\}$ is the vertex-set of an $f_{m\times n}$, where $d(u_i) = 4$ and $d(v_i^j) = 2$ and $E(f_{m\times n}) = \{u^1u^m, u^ju^{j+1} | 1 \leq j \leq m-1\} \cup \{v_i^jv_{i+1}^j | 1 \leq j \leq m, 1 \leq i \leq n-3\} \cup \{u^jv_1^j | 1 \leq j \leq m\} \cup \{u^1v_{n-2}^m, u^jv_{n-2}^{j-1} | 2 \leq j \leq m\}$ is the edge-set of an $f_{m\times n}$. We will complete a metric dimension of a flower graph $f_{m\times n}$ for any integer $n \geq 3$ and $m \in \{3, 4\}$ by the following proposition.

Proposition 2.1. *The metric dimension of a flower graph $f_{m\times n}$ for $m \in [3, 4]$ is as follow.*

$$dim(f_{m\times n}) = \begin{cases} 2 & \text{for } m = 3 \textit{ and } n \geq 3, \\ 3 & \text{for } m = 4 \textit{ and } n \geq 4. \end{cases}$$

Proof We first consider a graph $f_{3\times n}$. Clearly that $dim(f_{3\times n}) \geq 2$. To proving $dim(f_{3\times n}) \leq 2$, we suppose $W_1 = \left\{v^1_{\frac{n-1}{2}}, v^2_{\frac{n-1}{2}}\right\}$ for odd n and $W_2 = \left\{v^1_{\frac{n-2}{2}}, v^2_{\frac{n}{2}}\right\}$ for even n. Thus, the representations of all vertices in $V(f_{3\times n})$ are below.

- For odd n. Let $a = \frac{n-1}{2}$, then

$$r(u^j|W_1) = \begin{cases} (a, a+1) & \text{for } j = 1, \\ (a, a) & \text{for } j = 2, \\ (a+1, a) & \text{for } j = 3, \end{cases}$$

$$r(v_i^1|W_1) = \begin{cases} (a-i, a+1+i) & \text{for } i = 1, 2, \ldots, \frac{n-3}{2}, \\ (i-a, 3a-i) & \text{for } i = \frac{n-1}{2}, \frac{n+1}{2}, \ldots, n-2, \end{cases}$$

$$r(v_i^2|W_1) = \begin{cases} (a+i, a-i) & \text{for } i = 1, 2, \ldots, \frac{n-1}{2}, \\ (3a+1-i, i-a) & \text{for } i = \frac{n+1}{2}, \frac{n+3}{2}, \ldots, n-2, \end{cases}$$

$$r(v_i^3|W_1) = \begin{cases} (a+1+i, a+i) & \text{for } i = 1, 2, \ldots, \frac{n-3}{2}, \\ (n-1, n-1) & \text{for } i = \frac{n-1}{2}, \\ (3a-i, 3a+1-i) & \text{for } i = \frac{n+1}{2}, \frac{n+3}{2}, \ldots, n-2. \end{cases}$$

- For even n. Let $b = \frac{n-2}{2}$, then

$$r(u^j|W_2) = \begin{cases} (b, b+1) & \text{for } j = 1, \\ (b+1, b+1) & \text{for } j = 2, \\ (b+1, b) & \text{for } j = 3, \end{cases}$$

$$r(v_i^1|W_2) = \begin{cases} (b-i, b+1+i) & \text{for } i = 1, 2, \ldots, \frac{n-2}{2}, \\ (i-b, 3b+2-i) & \text{for } i = \frac{n}{2}, \frac{n+2}{2}, \ldots, n-2, \end{cases}$$

$$r(v_i^2|W_2) = \begin{cases} (b+1+i, b+1-i) & \text{for } i = 1, 2, \ldots, \frac{n-2}{2}, \\ (3b+2-i, i-b-1) & \text{for } i = \frac{n}{2}, \frac{n+2}{2}, \ldots, n-2, \end{cases}$$

$$r(v_i^3|W_2) = \begin{cases} (b+1+i, b+i) & \text{for } i = 1, 2, \ldots, \frac{n-2}{2}, \\ (3b+1-i, 3b+2-i) & \text{for } i = \frac{n}{2}, \frac{n+2}{2}, \ldots, n-2. \end{cases}$$

We can see that every vertex of $f_{3\times n}$ has distinct representation. Therefore, $2 \leq dim(f_{3\times n}) \leq 2$. Thus, $dim(f_{3\times n}) = 2$.

Next, we will prove that $dim(f_{4\times n}) = 3$. First, we show that $dim(f_{4\times n}) \geq 3$. Let W be a resolving set of $f_{4\times n}$. In this case, suppose that $dim(f_{4\times n}) = 2$, then there are the following possibilities to be discussed.

- Both vertices are in the inner cycle, C_4. In this case, suppose that $W = \{u^k, u^l\}$ is the resolving set of $f_{4\times n}$, where $k, l \in [1, 4]$. There are two possibilities about this resolving set, i.e. either $u^l \in N(u^k)$ or $u^l \notin N(u^k)$. If $u^l \in N(u^k)$, then we have $r(v_{n-2}^{k-1}|W) = r(v_1^k|W) = (1, 2)$. Note that $k-1 = 4$ if $k = 1$, $k+1 = 1$ if $k = 4$, $k+2 = 1$ if $k = 3$, and $k+2 = 2$ if $k = 4$. In the other hand, if $u^l \notin N(u^k)$, then $r(v_{n-2}^k|W) = r(v_1^l|W) = (2, 2)$.

- Both vertices belong to the set of outer vertices, C_n. In this case, suppose that $W = \{v_i^k, v_j^l\}$, where $k, l \in [1, 4]$ and $i, j \in [1, n-2]$. There are two possibilities about this resolving set, i.e. either $k = l$ or $k \neq l$. If $k = l$ then we get $r(v_1^{k-1}|W) = r(v_{n-2}^{k+2}|W) = (2+d(u^k, v_i^k), 2+d(u^k, v_j^l))$. If $k \neq l$, then $1 \leq |k-l| \leq 2$. For $|k-l| = 1$, w.l.o.g. we suppose that $l = k+1$ and $W = \{v_i^k, v_j^l\}$. There exist two vertices v_1^{k-1} and v_{n-2}^{l+1} such that $r(v_1^{k-1}|W) = r(v_{n-2}^{l+1}|W) = (1 + d(u^{k-1}, v_i^k), 1 + d(u^{k-1}, v_j^l))$. Now, we consider $|k-l| = 2$. Let $W = \{v_i^k, v_j^l\}$ be a resolving set. If $1 \leq i, j \leq \lfloor \frac{n-2}{2} \rfloor$, then there exist two vertices u^{k+1} and u^{l+1} such that $r(u^{k+1}|W) = r(u^{l+1}|W) = (1+d(u^k, v_i^k), 1+d(u^l, v_j^l))$. If $1 \leq i \leq \lfloor \frac{n-2}{2} \rfloor$ and $\lceil \frac{n}{2} \rceil \leq j \leq n-2$, then there exist two vertices u^l and v_1^{l+1} such that $r(u^l|W) = r(v_1^{l+1}|W) = (2 + d(u^k, v_i^k), 1 + d(u^{l+1}, v_j^l))$. If $\lceil \frac{n}{2} \rceil \leq i \leq n-2$ and $1 \leq j \leq \lfloor \frac{n-2}{2} \rfloor$ then $r(u^{l+1}|W) = r(v_{n-2}^{k+1}|W) = (2+d(u^{k+1}, v_i^k), 1+d(u^l, v_j^l))$. If $\lceil \frac{n}{2} \rceil \leq i, j \leq n-2$ then $r(u^k|W) = r(u^l|W) = (1+d(u^{k+1}, u_i^k), 1+d(u^{l+1}, v_j^l))$. If $1 \leq i \leq \lfloor \frac{n-2}{2} \rfloor$ and $j = \frac{n-1}{2}$ for odd n, then $r(v_{\frac{n-1}{2}}^{k-1}|W) = r(v_{\frac{n-3}{2}}^{k+1}|W) = (\frac{n-1}{2} + d(u^k, u_i^k), n-1)$. Next, for integer $i, j = \frac{n-1}{2}$, there exist two vertices u^k and u^{k+1} such that $r(u^k|W) = r(u^{k+1}|W) = (\frac{n-1}{2}, \frac{n+1}{2})$.

- One vertex is in the inner cycle and the other belongs to the set of outer cycles. For $i \in [1, n-2]$, we obtain the following: if $W = \{u^k, v_i^k\}$ then $r(v_{n-2}^{k+2}|W) = r(v_1^{k-1}|W) = (2, 2+d(u^k, v_i^k))$, if $W = \{u^k, v_i^l\}$ for $l = k+1$ or $l = k+2$ then

$r(v_1^k|W) = r(v_{n-2}^{k-1}|W) = (1, 1+d(u^k, v_i^l))$, and if $W = \{u^k, v_i^{k-1}\}$ then $r(v_{n-2}^k|W) = r(v_1^{k+1}|W) = (2, 2+d(u^k, v_i^{k-1}))$.

All cases would contradict the fact that W is a resolving set. Hence in all possible cases, there is no resolving set with two vertices for $V(f_{4\times n})$ thus implying that $dim(f_{4\times n}) \geq 3$, which completes the proof.
Next, we will show that $dim(f_{4\times n}) \leq 3$, for $n \geq 4$. Suppose that $W_1 = \left\{v^1_{\frac{n-1}{2}}, v^3_{\frac{n-1}{2}}, v^4_{\frac{n-1}{2}}\right\}$ for odd n and $W_2 = \left\{v^1_{\frac{n}{2}}, v^3_{\frac{n-2}{2}}, v^4_{\frac{n-2}{2}}\right\}$ for even n. Thus, the representations of all vertices in $V(f_{4\times n})$ are as follows.

- For odd n. Let $a = \frac{n-1}{2}$, then

$$r(u^j|W_1) = \begin{cases} (a, a+1, a-1+j) & \text{for } j = 1, 2, \\ (a+1, a, a+4-j) & \text{for } j = 3, 4, \end{cases}$$

$$r(v_i^1|W_1) = \begin{cases} (a-i, a+1+i, a+i) & \text{for } 1 \leq i \leq \frac{n-1}{2}, \\ (i-a, 3a+1-i, 3a+1-i) & \text{for } \frac{n+1}{2} \leq i \leq n-2, \end{cases}$$

$$r(v_i^2|W_1) = \begin{cases} (a+i, a+1+i, a+1+i) & \text{for } 1 \leq i \leq \frac{n-3}{2}, \\ (a+i, a+i, a+1+i) & \text{for } i = \frac{n-1}{2}, \\ (3a+1-i, 3a-i, 3a+1-i) & \text{for } \frac{n+1}{2} \leq i \leq n-2, \end{cases}$$

$$r(v_i^3|W_1) = \begin{cases} (a+1+i, a-i, a+1+i) & \text{for } 1 \leq i \leq \frac{n-3}{2}, \\ (a+1+i, a-i, a+i) & \text{for } i = \frac{n-1}{2}, \\ (3a-i+1, i-a, 3a-i) & \text{for } \frac{n+1}{2} \leq i \leq n-2, \end{cases}$$

$$r(v_i^4|W_1) = \begin{cases} (a+i+1, a+i, a-i) & \text{for } 1 \leq i \leq \frac{n-3}{2}, \\ (a+i, a+i, a-i) & \text{for } i = \frac{n-1}{2}, \\ (3a-i, 3a-i+1, i-a) & \text{for } \frac{n+1}{2} \leq i \leq n-2. \end{cases}$$

- For even n. Let $b = \frac{n}{2}$, then

$$r(u^j|W_2) = \begin{cases} (b-j+1, b-j+1, b+j-1) & \text{for } j = 1, 2, \\ (b+j-3, b+j-4, b-j+3) & \text{for } j = 3, 4, \end{cases}$$

$$r(v_i^1|W_2) = \begin{cases} (b-i, b+i+1, b+i) & \text{for } 1 \leq i \leq \frac{n-2}{2}, \\ (i-b, 3b-1-i, 3b-i) & \text{for } \frac{n}{2} \leq i \leq n-2, \end{cases}$$

$$r(v_i^2|W_2) = \begin{cases} (b-1+i, b+i, b+i+1) & \text{for } 1 \leq i \leq \frac{n-2}{2}, \\ (3b\text{-}1-i, 3b-2-i, 3b-1-i) & \text{for } \frac{n}{2} \leq i \leq n-2, \end{cases}$$

$$r(v_i^3|W_2) = \begin{cases} (b+i, b-i-1, b+i) & \text{for } 1 \le i \le \frac{n-2}{2}, \\ (3b-i, i-b+1, 3b-i-2) & \text{for } \frac{n}{2} \le i \le n-2, \end{cases}$$

$$r(v_i^4|W_2) = \begin{cases} (b+i+1, b+i, b-i-1) & \text{for } 1 \le i \le \frac{n-2}{2}, \\ (3b-i-1, 3b-i, i-b+1) & \text{for } \frac{n}{2} \le i \le n-2. \end{cases}$$

Since the representation of every vertex of $f_{4\times n}$ with respect to W is different, the set W is a resolving partition. Hence, $dim(f_{4\times n}) \le 3$ and we conclude that $dim(f_{4\times n}) = 3$.
As an illustration, a metric dimension of $f_{3\times 3}$, $f_{3\times 4}$, and $f_{4\times 4}$ is depicted in Figure 1, where the representasion set is bold and diamond vertices (and bold triangle vertex). According

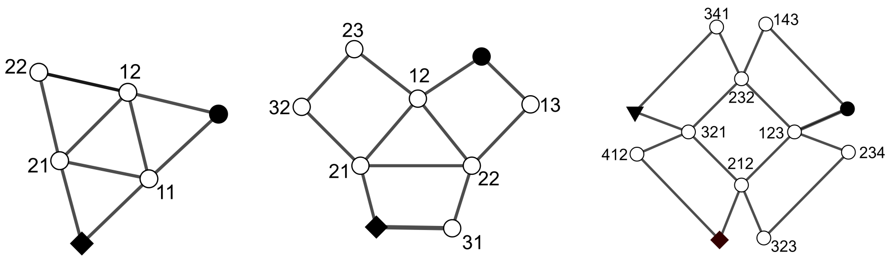

Figure 1: A metric dimension of $f_{3\times 3}$, $f_{3\times 4}$ and $f_{4\times 4}$.

to Theorem 1.5 and Proposition 2.1, there are different results for $dim(f_{3\times 3})$. However, this is clear by Figure 2.1.

Proposition 2.2. *Let $m \ge 3$ be an integer. Then $pd(f_{m\times 3}) = 3$.*

Proof Suppose $V(f_{m\times 3}) = \{u_i, v_i | 1 \le i \le m\}$, where $d(u_i) = 4$ and $d(v_i) = 2$ for each $i \in [1, m]$. To prove that $pd(f_{m\times 3}) \le 3$, define an ordered 3-partition $\Lambda = \{S_1, S_2, S_3\}$ of $V(f_{m\times 3})$ as follows.

- For $m = 0 \pmod 3$,

$$\begin{aligned} S_1 &= \left\{u_i, v_i | 1 \le i \le \tfrac{m}{3}\right\}, \\ S_2 &= \left\{u_i, v_i | \tfrac{m}{3} + 1 \le i \le m - \tfrac{m}{3}\right\}, \\ S_3 &= \left\{u_i, v_i | m + 1 - \tfrac{m}{3} \le i \le m\right\}. \end{aligned}$$

The representation of vertices of $V(f_{m\times 3})$ with respect to an ordered 3- partition Λ are below.

$$r(u_i|\Lambda) = \begin{cases} \left(0, \frac{m}{3} + 1 - i, i\right) & \text{for } 1 \le i \le \frac{m}{3}, \\ \left(i - \frac{m}{3}, 0, 2\left(\frac{m}{3}\right) + 1 - i\right) & \text{for } \frac{m}{3} + 1 \le i \le m - \frac{m}{3}, \\ \left(m + 1 - i, i - 2\left(\frac{m}{3}\right), 0\right) & \text{for } m + 1 - \frac{m}{3} \le i \le m, \end{cases}$$

$$r(v_i|\Lambda) = \begin{cases} \left(0, \frac{m}{3}+2-i, 1+i\right) & \text{for } 1 \le i \le \frac{m}{3}, \\ \left(i-\frac{m}{3}+1, 0, 2\left(\frac{m}{3}\right)+1-i\right) & \text{for } \frac{m}{3}+1 \le i \le m-\frac{m}{3}, \\ \left(m+1-i, i-2\left(\frac{m}{3}\right)+1, 0\right) & \text{for } m+1-\frac{m}{3} \le i \le m. \end{cases}$$

- For $m \neq 0 \pmod 3$,

$$\begin{aligned} S_1 &= \left\{u_i, v_i | 1 \le i \le \left\lfloor \tfrac{m}{3} \right\rfloor\right\} \cup \left\{u_{\lceil \frac{m}{3} \rceil}\right\}, \\ S_2 &= \left\{u_i, v_i | \left\lceil \tfrac{m}{3} \right\rceil + 1 \le i \le m - \left\lceil \tfrac{m}{3} \right\rceil\right\} \cup \left\{u_{m-\lfloor \frac{m}{3} \rfloor}, v_{\lceil \frac{m}{3} \rceil}\right\}, \\ S_3 &= \left\{u_i, v_i | m+1-\left\lfloor \tfrac{m}{3} \right\rfloor \le i \le m\right\} \cup \left\{v_{m-\lfloor \frac{m}{3} \rfloor}\right\}. \end{aligned}$$

The representation of vertices of $V(f_{m\times 3})$ with respect to an ordered 3- partition Λ are below.

$$r(u_i|\Lambda) = \begin{cases} \left(0, \left\lceil \frac{m}{3} \right\rceil + 1 - i, i\right) & \text{for } 1 \le i \le \left\lceil \frac{m}{3} \right\rceil, \\ \left(i-\left\lceil \frac{m}{3} \right\rceil, 0, 2\left\lceil \frac{m}{3} \right\rceil - i\right) & \text{for } \left\lceil \frac{m}{3} \right\rceil + 1 \le i \le m-\left\lfloor \frac{m}{3} \right\rfloor, \\ \left(m+1-i, i-\left\lceil \frac{m}{3} \right\rceil - \left\lfloor \frac{m}{3} \right\rfloor, 0\right) & \text{for } m+1-\left\lfloor \frac{m}{3} \right\rfloor \le i \le m, \end{cases}$$

$$r(v_i|\Lambda) = \begin{cases} \left(0, \left\lceil \frac{m}{3} \right\rceil + 1 - i, 1+i\right) & \text{for } 1 \le i \le \left\lfloor \frac{m}{3} \right\rfloor, \\ \left(1-\left\lceil \frac{m}{3} \right\rceil + i, 0, m+1-\left\lfloor \frac{m}{3} \right\rfloor - i\right) & \text{for } \left\lceil \frac{m}{3} \right\rceil \le i \le m-\left\lceil \frac{m}{3} \right\rceil, \\ \left(m+1-i, 1-\left\lceil \frac{m}{3} \right\rceil - \left\lfloor \frac{m}{3} \right\rfloor + i, 0\right) & \text{for } m-\left\lfloor \frac{m}{3} \right\rfloor \le i \le m. \end{cases}$$

We can see that every vertex of $V(f_{m\times 3})$ has distinct representation. Therefore, $pd(f_{m\times 3}) \le 3$. Next, by Theorem 1.3, we can conclude that $pd(f_{m\times 3}) = 3$.

Proposition 2.3. *Let $n \ge 4$ be an integer. If $m \in \{3, 4, 5\}$, then $pd(f_{m\times n}) = 3$.*

Proof By Theorem 1.3, we obtain $pd(f_{m\times n}) \ge 3$. To prove the upper bound, we break up the proof into three cases.

(i) For $m = 3$. We define an ordered 3-partition $\Lambda = \{S_1, S_2, S_3\}$ of $V(f_{3\times n})$, where $S_j = \{u^j, v_i^j\}$, for $1 \le i \le n-2$ and $j \in [1, 3]$. So, the vertex representation of each vertex of $V(f_{3\times n})$ is as follows.

$$r(u^j|\Lambda) = \begin{cases} (0,1,1) & \text{for } j = 1, \\ (1,0,1) & \text{for } j = 2, \\ (1,1,0) & \text{for } j = 3, \end{cases}$$

For $1 \le i \le \left\lfloor \frac{n-2}{2} \right\rfloor$,

$$r(v_i^j|\Lambda) = \begin{cases} (0, 1+i, 1+i) & \text{for } j = 1, \\ (1+i, 0, 1+i) & \text{for } j = 2, \\ (1+i, 1+i, 0) & \text{for } j = 3, \end{cases}$$

For $\lfloor \frac{n}{2} \rfloor \leq i \leq n-2$.

$$r(v_i^j|\Lambda) = \begin{cases} (0, n-1-i, n-i) & \text{for } j=1, \\ (n-i, 0, n-1-i) & \text{for } j=2, \\ (n-1-i, n-i, 0) & \text{for } j=3. \end{cases}$$

(ii) For $m = 4$. To show that $pd(f_{4\times n}) \leq 3$, we consider two cases.

- For $n = 4$. We set an ordered 3-partition $\Lambda = \{S_1, S_2, S_3\}$ of $V(f_{4\times 4})$, where $S_1 = \{u^1, u^2, v_1^1, v_2^1\}$, $S_2 = \{u^3, v_1^2, v_2^2, v_1^3\}$, and $S_3 = \{u^4, v_2^3, v_1^4, v_2^4\}$. The representation of each vertex is as follows.

i	$r(u^i\|\Lambda)$	$r(v_1^i\|\Lambda)$	$r(v_2^i\|\Lambda)$
1	$(0,2,1)$	$(0,3,2)$	$(0,2,3)$
2	$(0,1,2)$	$(1,0,3)$	$(2,0,2)$
3	$(1,0,1)$	$(2,0,1)$	$(2,1,0)$
4	$(1,1,0)$	$(2,2,0)$	$(1,3,0)$

- For $n \geq 5$. We define an ordered 3-partition $\Lambda = \{S_1, S_2, S_3\}$ of $V(f_{4\times n})$ as follows.

$$\begin{aligned} S_1 &= \{u^1, u^2\} \cup \{v_1^1, v_2^1, \ldots, v_{n-2}^1\} \cup \{v_2^2, v_3^2, \ldots, v_{n-3}^2\}, \\ S_2 &= \{u^3\} \cup \{v_1^2, v_{n-2}^2\} \cup \{v_1^3, v_2^3, \ldots, v_{n-2}^3\}, \\ S_3 &= \{u^4\} \cup \{v_1^4, v_2^4, \ldots, v_{n-2}^4\}. \end{aligned}$$

We obtain the representation of each vertex of $V(f_{4\times n})$ as follows.

$$r(u^j|\Lambda) = \begin{cases} (0, 3-j, j), & \text{for } j=1,2, \\ (1,0,1), & \text{for } j=3, \\ (1,1,0), & \text{for } j=4, \end{cases}$$

$$r(v_i^j|\Lambda) = \begin{cases} (0, \alpha_j, \beta_j), & \text{for } j=1,2, \\ (\alpha_j, 0, \beta_j), & \text{for } j=3, \\ (\alpha_j, \beta_j, 0), & \text{for } j=4, \end{cases}$$

where

$$\alpha_1 = \begin{cases} 2+i, & \text{for } i = 1,2,\ldots,\lceil \frac{n-4}{2} \rceil, \\ n-i, & \text{for } i = \lceil \frac{n-2}{2} \rceil, \lceil \frac{n}{2} \rceil, \ldots, n-2, \end{cases}$$

$$\alpha_2 = \begin{cases} 1, & \text{for } i = 1, n-2, \\ i-1, & \text{for } i = 2,3,\ldots,\lceil \frac{n-3}{2} \rceil, \\ n-2-i, & \text{for } i = \lceil \frac{n-1}{2} \rceil, \lceil \frac{n+1}{2} \rceil, \ldots, n-3, \end{cases}$$

$$\alpha_3 = \begin{cases} 1+i, & \text{for } i = 1,2,\ldots,\lceil \frac{n-3}{2} \rceil, \\ n-i, & \text{for } i = \lceil \frac{n-1}{2} \rceil, \lceil \frac{n+1}{2} \rceil, \ldots, n-2, \end{cases}$$

$$\alpha_4 = \begin{cases} 1+i, & \text{for } i = 1,2,\ldots,\lceil \frac{n-3}{2} \rceil, \\ n-1-i, & \text{for } i = \lceil \frac{n-1}{2} \rceil, \lceil \frac{n+1}{2} \rceil, \ldots, n-2, \end{cases}$$

$$\beta_1 = \begin{cases} 1+i, & \text{for } i = 1,2,\ldots,\lceil \frac{n-1}{2} \rceil, \\ n+1-i, & \text{for } i = \lceil \frac{n+1}{2} \rceil, \lceil \frac{n+3}{2} \rceil, \ldots, n-1, \end{cases}$$

$$\beta_2 = \begin{cases} 3, & \text{for } i = 1, \\ 2+i, & \text{for } i = 2,3,\ldots,\lceil \frac{n-4}{2} \rceil, \\ n-i, & \text{for } i = \lceil \frac{n-2}{2} \rceil, \lceil \frac{n}{2} \rceil, \ldots, n-3, \\ 2, & \text{for } i = n-2, \end{cases}$$

$$\beta_3 = \begin{cases} 1+i, & \text{for } i = 1,2,\ldots,\lceil \frac{n-3}{2} \rceil, \\ n-1-i, & \text{for } i = \lceil \frac{n-1}{2} \rceil, \lceil \frac{n+1}{2} \rceil, \ldots, n-2, \end{cases}$$

$$\beta_4 = \begin{cases} 1+i, & \text{for } i = 1,2,\ldots,\lceil \frac{n-2}{2} \rceil, \\ n+1-i, & \text{for } i = \lceil \frac{n}{2} \rceil, \lceil \frac{n+2}{2} \rceil, \ldots, n-2. \end{cases}$$

(iii) For $m = 5$. We set an ordered 3-partition $\Lambda = \{S_1, S_2, S_3\}$ of $V(f_{5\times n})$ as follows.

$$\begin{aligned} S_1 &= \{u^1, u^2\} \cup \left\{v_1^1, v_2^1, \ldots, v_{n-2}^1\right\} \cup \left\{v_2^2, v_3^2, \ldots, v_{n-2}^2\right\}, \\ S_2 &= \{u^3, u^4, v_1^2\} \cup \left\{v_1^3, v_2^3, \ldots, v_{n-2}^3\right\} \cup \left\{v_i^4 | i \in [\lceil \tfrac{n-2}{2} \rceil, n-3]\right\}, \\ S_3 &= \left\{u^5, v_{n-2}^4\right\} \cup \left\{v_1^4, v_2^4, \ldots, v_{\lceil \frac{n-4}{2} \rceil}^4\right\} \cup \{v_i^5 | i \in [1, n-2]\}. \end{aligned}$$

Then, each vertex of $V(f_{5\times n})$ has the representation as follows.

$$r(u^j|\Lambda) = \begin{cases} (0, 3-j, j), & \text{for } j = 1,2, \\ (j-2, 0, 5-j), & \text{for } j = 3,4, \\ (1,1,0), & \text{for } j = 5, \end{cases}$$

$$r(v_i^1|\Lambda) = \begin{cases} (0, 2+i, 1+i), & \text{for } i = 1,2,\ldots,\lceil \frac{n-3}{2} \rceil, \\ (0, \frac{n+1}{2}, \frac{n+1}{2}), & \text{for } i = \frac{n-1}{2}, \\ (0, n-i, n+1-i), & \text{for } i = \lceil \frac{n}{2} \rceil, \lceil \frac{n+2}{2} \rceil, \ldots, n-2, \end{cases}$$

$$r(v_i^2|\Lambda) = \begin{cases} (1,0,3), & \text{for } i = 1, \\ (0, i-1, 2+i), & \text{for } i = 2,3,\ldots,\lfloor \frac{n-1}{2} \rfloor, \\ (0, n-1-i, n+1-i), & \text{for } i = \lfloor \frac{n+1}{2} \rfloor, \lfloor \frac{n+3}{2} \rfloor, \ldots, n-2, \end{cases}$$

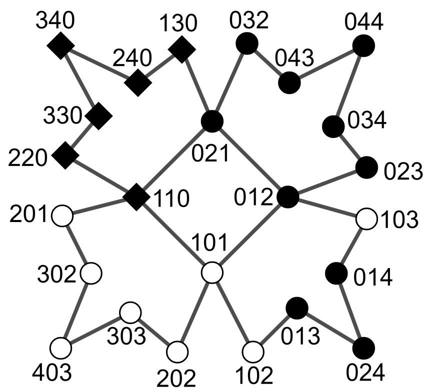

Figure 2: A partition dimension of $f_{4\times7}$ is 3

$$r(v_i^3|\Lambda) = \begin{cases} (1+i, 0, 2+i), & \text{for } i = 1, 2, \ldots, \lceil\frac{n-3}{2}\rceil, \\ (\frac{n+1}{2}, 0, \frac{n+1}{2}), & \text{for } i = \frac{n-1}{2}, \\ (n+1-i, 0, n-i), & \text{for } i = \lceil\frac{n}{2}\rceil, \lceil\frac{n+2}{2}\rceil, \ldots, n-2, \end{cases}$$

$$r(v_i^4|\Lambda) = \begin{cases} (2+i, i, 0), & \text{for } i = 1, 2, \ldots, \lceil\frac{n-4}{4}\rceil, \\ (2+i, \lceil\frac{n-2}{2}\rceil - i, 0), & \text{for } i = \lceil\frac{n}{4}\rceil, \lceil\frac{n+2}{4}\rceil, \ldots, \lceil\frac{n-4}{2}\rceil, \\ (n-i, 0, i - \lceil\frac{n-4}{2}\rceil), & \text{for } i = \lceil\frac{n-2}{2}\rceil, \lceil\frac{n}{2}\rceil, \ldots, \lceil\frac{3n-10}{4}\rceil, \\ (n-i, 0, n-2-i), & \text{for } i = \lceil\frac{3n-6}{4}\rceil, \lceil\frac{3n-2}{4}\rceil, \ldots, n-3, \\ (2, 1, 0), & \text{for } i = n-2, \end{cases}$$

$$r(v_i^5|\Lambda) = \begin{cases} (1+i, 1+i, 0), & \text{for } i = 1, 2, \ldots, \lceil\frac{n-3}{2}\rceil, \\ (\lfloor\frac{n}{2}\rfloor, \lfloor\frac{n+2}{2}\rfloor, 0), & \text{for } i = \frac{n-1}{2}, \\ (n-1-i, n+1-i, 0), & \text{for } i = \lceil\frac{n}{2}\rceil, \lceil\frac{n+2}{2}\rceil, \ldots, n-2. \end{cases}$$

From all cases, we can see that every vertex of $V(f_{m\times n})$, $m \in \{3, 4, 5\}$, has distinct representation.

As an illustration, Figure 2 shows the partition dimension of the flower graph $f_{4\times7}$, where the first partition is represented by the black vertices, the second one by the white vertices, and the remaining by the diamond vertices.

2.2 Pencil graph

Here, we give the exact value of the metric and partition dimension of a pencil graph Pc_m. A pencil graph, denoted by Pc_m, is a 3-regular connected graph on $2m+2$ vertices and $3m+3$ edges which is having the set of vertices

$$V(Pc_m) = \{u_0, u_1, \ldots, u_m\} \cup \{v_0, v_1, \ldots, v_m\}$$

and the set of edges

$$E(Pc_m) = \{u_iu_{i+1}, v_iv_{i+1}|i \in [1, m-1]\} \cup \{u_iv_i|i \in [1, m]\} \cup \{u_0u_1, u_0v_1, v_0u_m, v_0v_m\}.$$

First, we prove that the metric dimension of a pencil graph Pc_m, by the following theorem.

Theorem 2.4. *Let $m \geq 2$ be an integer. The metric diension of a pencil graph Pc_m is as follows.*

$$dim(Pc_m) = \begin{cases} 2 & \text{if } m \text{ even or } m = 3, \\ 3 & \text{if } m \geq 5 \text{ odd.} \end{cases}$$

Proof Based on Theorem 1.1, we have $dim(Pc_m) \geq 2$. Next, we will show that $dim(Pc_m) \leq 2$, by constructing a resolving set W, where $|W| = 2$, such that the representation of each vertex of the $Pc_m \backslash W$ is distinct. We break up the proof into three cases.

- For $m = 3$. The metric dimension of Pc_3 is given by Figure 3. The resolving set W is stated by he black and diamond vertex, namely $W = \{$black vertex, diamond vertex$\}$.

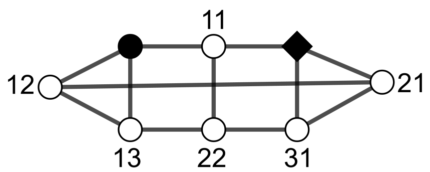

Figure 3: $dim(Pc_3) = 2$

- For m even. We consider $W = \{u_0, u_{\frac{m}{2}}\}$. The representations of each vertex of $V(Pc_m) \backslash \{u_0, u_{\lfloor \frac{m}{2} \rfloor}\}$ are described below.

$$r(u_i|W) = \begin{cases} \left(i, \frac{m}{2} - i\right), & \text{for } i = 1, 2, \ldots, \frac{m}{2} - 1, \\ \left(\frac{2m+4}{2} - i, i - \frac{m}{2}\right), & \text{for } i = \frac{m+2}{2}, \frac{m+4}{2}, \ldots, m, \end{cases}$$

$$r(v_i|W) = \begin{cases} \left(1, \frac{m+2}{2}\right) & \text{for } i = 0, \\ \left(i, \frac{m+2}{2} - i\right) & \text{for } i = 1, 2, \ldots, \frac{m}{2}, \\ \left(\frac{2m+4}{2} - i, i - \frac{m-2}{2}\right) & \text{for } i = \frac{m}{2}, \frac{m+2}{2}, \ldots, m. \end{cases}$$

- For $m \geq 5$ odd. First, we show that every subset of $V(G)$ with two vertices is not a resolving set. Without loss of generality, we consider four cases, namely $W = \{u_0, v_0\}$, $W = \{u_0, u_i\}$, $W = \{u_i, u_j\}$, or $W = \{u_i, v_j\}$ for some $i, j \in [1, m]$.

First, suppose $W = \{u_0, v_0\}$. In this case, every pair u_i dan v_i for all $i \in [1, m]$ will have the same representasion. We suppose now $W = \{u_0, u_i\}$. For all $i \in [1, \frac{m+1}{2}]$, W is not resolving set, since the vertices $u_{\frac{m+3}{2}}$ and $v_{\frac{m+1}{2}}$ have the same representation, namely $(\frac{m+1}{2}, \frac{m+3}{2} - i)$. On the other hand, the vertices $u_{\frac{m+1}{2}}$ and $v_{\frac{m+3}{2}}$ have the same representation, namely $(\frac{m+1}{2}, i - \frac{m+1}{2})$ when $i \in [\frac{m+3}{2}, m]$. Furthermore, we consider $W = \{u_i, u_j\}$. For some $i, j \in [1, \frac{m+1}{2}]$, we obtain $r(u_0|W) = r(v_1|W) = (i, j)$; for some $i, j \in [1, m-1]$ satisfying $1 \leq j - i \leq \frac{m-1}{2}$, two distinct vertices u_{j+1} and v_j have the representation $(j+1-i, 1)$; for $i, j \in [1, m]$ satisfying $\frac{m+3}{2} \leq j - i \leq m - 1$, we have $r(u_{i+\frac{m+1}{2}}|W) = r(v_{i+\frac{m+3}{2}}|W) = (\frac{m+1}{2}, j - i - \frac{m+1}{2})$; for some $i, j \in [1, m-1]$ satisfying $j - i = \frac{m+1}{2}$, two distinct vertices v_{j-1} and v_{j+1} have the representation $(\frac{m+1}{2}, 2)$. The last case, we consider $W = \{u_i, v_j\}$. For each $i = j$ and $i \in [2, m-1]$, two distinct vertices u_{i-1} and u_{i+1} have representation $(1, 2)$. When $W = \{u_1, v_1\}$, we obtain $r(u_m|W) = r(v_m|W) = (3, 3)$. For $i < j$ and $1 \leq j - i \leq \frac{m+1}{2}$, we obtain $r(u_{i+1}|W) = r(v_i|W) = (1, j - i)$. For $i < j$ and $\frac{m+3}{2} \leq j - i \leq m - 1$, we get $r(u_{\frac{m+1}{2}}|W) = r(v_{\frac{m-1}{2}}|W) = (\frac{m+1}{2} - i, j - \frac{m-1}{2})$. Therefore every possible set with two vertices is not a resolving set. So, $dim(Pc_m) \geq 3$.

Now, we prove that $dim(Pc_m) \leq 3$ by taking the set $W = \{u_0, u_1, u_{\frac{m-1}{2}}\}$. This set gave the representation of each vertex of $V(Pc_m)\backslash W$ is below.

$$r(u_i|W) = \begin{cases} \left(i, i-1, \frac{m-1}{2} - i\right), & \text{for } i = 2, 3, \ldots, \frac{m-3}{2}, \\ \left(\frac{m+1}{2}, i-1, i - \frac{m-1}{2}\right), & \text{for } i = \frac{m+1}{2}, \frac{m+3}{2}, \ldots, \frac{m+3}{2}, \\ \left(m+2-i, m+3-i, i - \frac{m-1}{2}\right), & \text{for } i = \frac{m+5}{2}, \frac{m+7}{2}, \ldots, m, \end{cases}$$

$$r(v_i|W) = \begin{cases} \left(1, 2, \frac{m+1}{2}\right), & \text{for } i = 0, \\ \left(i, i, \frac{m+1}{2} - i\right), & \text{for } i = 1, 2, \ldots, \frac{m-1}{2}, \\ \left(\frac{m+1}{2}, \frac{m+1}{2}, 2\right), & \text{for } i = \frac{m+1}{2}, \\ \left(m+2-i, m+3-i, i - \frac{m-3}{2}\right), & \text{for } i = \frac{m+3}{2}, \frac{m+5}{2}, \ldots, m. \end{cases}$$

From all cases, we can see that every vertex of $V(Pc_m)$ has distinct representation with respect to the set W. Hence, $dim(Pc_m) \leq 3$. Therefore, $dim(Pc_m) = 3$, for each odd positive integer $m \geq 5$.

Next, we show that the partition dimension of a pencil graph Pc_m for even m is one more than its metric dimension.

Theorem 2.5. *Let $m \geq 2$ be an integer. The partition dimension of a pencil graph Pc_m is three for any integer $m \geq 2$, i.e., $pd(Pc_m) = 3$.*

Proof According to Theorem 1.3, $pd(Pc_m) \geq 3$. Next, we will show that $pd(Pc_m) \leq 3$, by constructing an ordered 3-partition $\Lambda = \{S_1, S_2, S_3\}$ such that the representation of each vertex of the Pc_m is distinct. We break up the proof into three cases.

- For $m \in \{2,3,5\}$. The partition dimension of Pc_2, Pc_3, and Pc_5, is given in Figure 4, where the first partition is given by the black vertices, the second one by the white partition, and the remaining by the diamond vertices.

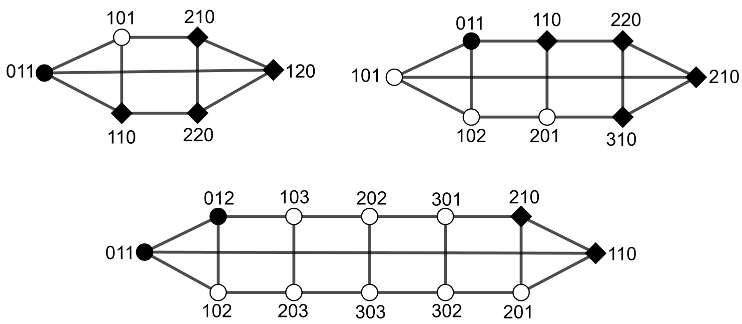

Figure 4: $pd(Pc_m) = 3$ for $m = 2, 3, 5$.

- For $m \geq 4$ even. Consider $S_1 = \{u_0\}$, $S_2 = \left\{u_{\frac{m}{2}}\right\}$, and $S_3 = V(Pc_m) \setminus \left\{u_0, u_{\frac{m}{2}}\right\}$. The representations of each vertex of $V(Pc_m)$ are as follows:

$$
\begin{aligned}
r\left(u_0|\Lambda\right) &= \left(0, \tfrac{m}{2}, 1\right),\\
r\left(u_{\frac{m}{2}}|\Lambda\right) &= \left(\tfrac{m}{2}, 0, 1\right),\\
r\left(u_i|\Lambda\right) &= \begin{cases} \left(i, \frac{m}{2} - i, 0\right) & \text{for } 1 \leq i \leq \frac{m}{2} - 1,\\ \left(m + 2 - i, i - \frac{m}{2}, 0\right) & \text{for } \frac{m}{2} + 1 \leq i \leq m, \end{cases}\\
r(v_i|\Lambda) &= \begin{cases} \left(\frac{m+2}{2}, 1, 0\right) & \text{for } i = 0,\\ \left(i, 1 + \frac{m}{2} - i, 0\right) & \text{for } 1 \leq i \leq \frac{m}{2},\\ \left(m + 2 - i, 1 - \frac{m}{2} + i, 0\right) & \text{for } \frac{m}{2} + 1 \leq i \leq m. \end{cases}
\end{aligned}
$$

- For $m \geq 7$ odd. Consider $S_1 = \{u_0, u_1\}$, $S_2 = \{u_{m-1}, u_m\}$, and $S_3 = V(Pc_m) \setminus \{S_1 \cup S_2\}$. So, the representations of each vertex of $V(Pc_m)$ are as follows.

$$
\begin{aligned}
r(u_0|\Lambda) &= (0,2,1),\\
r(u_1|\Lambda) &= (0,3,1),\\
r(u_{m-1}|\Lambda) &= (3,0,1),\\
r(u_m|\Lambda) &= (2,0,1),\\
r(u_i|\Lambda) &= \begin{cases} (i-1, i+2, 0), & \text{for } i = 2,3,\dots,\frac{m-3}{2},\\ \left(\frac{m-3}{2}, \frac{m-1}{2}, 0\right), & \text{for } i = \frac{m-1}{2},\\ \left(\frac{m-1}{2}, \frac{m-3}{2}, 0\right), & \text{for } i = \frac{m+1}{2},\\ (m+2-i, m-1-i, 0), & \text{for } i = \frac{m+3}{2}, \frac{m+5}{2}, \dots, m-2, \end{cases}\\
r(v_i|\Lambda) &= \begin{cases} (i, i+2, 0), & \text{for } i = 1,2,\dots,\frac{m-3}{2},\\ \left(\frac{m-1}{2}, \frac{m+1}{2}, 0\right), & \text{for } i = \frac{m-1}{2},\\ \left(\frac{m+1}{2}, \frac{m-1}{2}, 0\right), & \text{for } i = \frac{m+1}{2},\\ (m+2-i, m-i, 0), & \text{for } i = \frac{m+3}{2}, \frac{m+5}{2}, \dots, m-1,\\ (2,1,0), & \text{for } i = m. \end{cases}
\end{aligned}
$$

From all cases, we can see that every vertex of $V(Pc_m)$ has distinct representation of an ordered 3-partition Λ. Hence, $pd(Pc_m) \leq 3$.

3 Concluding Remark

This research completely the problem of the metric dimension of the flower graph $f_{m\times n}$ namely

$$
dim(f_{m\times n}) = \begin{cases} 2 & \text{for } m \geq 4 \text{ even and } n = 3 \text{ [15]},\\ 3 & \text{for } m = 4 \text{ and } n \geq 4,\\ 2 & \text{for } m = 3 \text{ and } n \geq 3,\\ 3 & \text{for } m \geq 5 \text{ odd and } n = 3 \text{ [15]},\\ \lceil \frac{m}{2} \rceil & \text{for } m \geq 5 \text{ and } n \geq 4 \text{ [15]}. \end{cases}
$$

However we still have the open problem of the partition dimension of a flower graph $f_{m\times n}$ for each integer $m \geq 6$ and $n \geq 4$. Meanwhile, the metric and partition dimensions of the pencil graph Pc_m are complete for every integer $m \geq 2$.

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