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4-Total Mean Cordial Labeling of Some Trees

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ABSTRACT

Let G be a graph. Let $f : V(G) \rightarrow \{0, 1, 2, \ldots, k-1\}$ be a function where $k \in \mathbb{N}$ and $k > 1$. For each edge uv, assign the label $f(uv) = \left\lceil \frac{f(u)+f(v)}{2} \right\rceil$. f is called a k-total mean cordial labeling of G if $|t_{mf}(i) - t_{mf}(j)| \leq 1$, for all $i, j \in \{0, 1, 2, \ldots, k-1\}$, where $t_{mf}(x)$ denotes the total number of vertices and edges labelled with x, $x \in \{0, 1, 2, \ldots, k-1\}$. A graph with admit a k-total mean cordial labeling is called k-total mean cordial graph. In this paper we examine the 4-Total mean cordial labeling of some trees.

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1 Introduction

In this paper, we consider simple, finite and undirected graphs only. Graph labeling is applied in several area of sciences like coding theory, x-ray crystallography, radar, astronomy, circuit design, communication network addressing, data base management, secret sharing schmes, cryptology, models for constraint programming over finite domains, etc [2]. Cordial labeling was introduced by Cahit [1]. Motivated by this labeling method the notion of

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k-total mean cordial labeling has been introduced in [5]. Also in [5, 6, 7, 8, 9, 10, 11, 12, 13] the 4-total mean cordial labeling behaviour of several graphs like cycle, complete graph, star, bistar, comb and crown have been investigated. In this paper we examine the 4-total mean cordial labeling of some trees. Let x be any real number. Then $\lceil x \rceil$ stands for the smallest integer greater than or equal to x. Terms are not defined here follow from Harary[3] and Gallian[2].

2 k-total mean cordial graph

Definition 2.1. Let G be a graph. Let $f : V(G) \rightarrow \{0, 1, 2, \ldots, k-1\}$ be a function where $k \in \mathbb{N}$ and $k > 1$. For each edge uv, assign the label $f(uv) = \left\lceil \frac{f(u)+f(v)}{2} \right\rceil$. f is called a k-total mean cordial labeling of G if $|t_{mf}(i) - t_{mf}(j)| \leq 1$, for all $i, j \in \{0, 1, 2, \ldots, k-1\}$, where $t_{mf}(x)$ denotes the total number of vertices and edges labelled with x, $x \in \{0, 1, 2, \ldots, k-1\}$. A graph with admit a k-total mean cordial labeling is called k-total mean cordial graph.

3 Preliminaries

Definition 3.1. [4] The Lilly graph I_n, $n \geq 2$ is constructed by two copy of the stars $2K_{1,n}$, $n \geq 2$ joining the two copy of the path graphs $2P_n$, $n \geq 2$ with sharing a common vertex. Let $V(I_n) = \{x_i, y_i : 1 \leq i \leq n\} \cup \{u_i : 1 \leq i \leq n\} \cup \{v_i : 1 \leq i \leq n-1\}$ and $E(I_n) = \{u_n x_i, u_n y_i : 1 \leq i \leq n\} \cup \{u_i u_{i+1} : 1 \leq i \leq n-1\} \cup$ $\{v_i v_{i+1} : 1 \leq i \leq n-2\} \cup \{u_n v_1\}$. Clearly $|V(I_n)| + |E(I_n)| = 8n - 3$.

Definition 3.2. [3] Let G_1, G_2 respectively be (p_1, q_1), (p_2, q_2) graphs. The *corona* of G_1 with G_2 is the graph $G_1 \odot G_2$ obtained by taking one copy of G_1 , p_1 copies of G_2 and joining the i^{th} vertex of G_1 by an edge to every vertex in the i^{th} copy of G_2 where $1 \leq i \leq p_1$.

Definition 3.3. [4] The banana tree $B(m, n)$ is a graph obtained by connecting one leaf of each of m-copies of the star $K_{1,n}$ with a single root vertex that is distinct from all the stars.

Definition 3.4. [4] The coconut tree $CT(m, n)$ is a graph obtained from the path P_n by appending m new pendent edges at an end vertex of P_n.

Definition 3.5. [3] The graph $G^{(n)}$, $n \geq 2$ is constructed by path and stars. Let the vertex and edge set of $G^{(n)}$ denote by $V\left(G^{(n)}\right) = \{u_i, v_i, w_i, w_{i,j} : 1 \leq i \leq n, 1 \leq j \leq n-1\}$ and $E\left(G^{(n)}\right)=\{u_iv_i, v_iw_i, w_iw_{i,j} : 1 \leq i \leq n, 1 \leq j \leq n-1\} \cup \{u_iu_{i+1} : 1 \leq i \leq n-1\}$ respectively.

4 Main results

Theorem 4.1. The Lilly graph I_n is a 4-total mean cordial for all values of $n \geq 2$.

Proof Take the vertex set and the edge set of I_n as in Definition 3.1.
Clearly $|V(I_n)| + |E(I_n)| = 8n - 3$.
Assign the label 0 to the n vertices $x_1, x_2, \ldots, x_n$. Now we assign the label 1 to the $n-1$ vertices $y_1, y_2, \ldots, y_{n-1}$. Now we assign the label 2 to the vertex u_1. Next we assign the label 3 to the $n-1$ vertices $u_2, u_3, \ldots, u_n$. Finally we assign the label 3 to the n vertices $v_1, v_2, \ldots, v_n$. Obviously $t_{mf}(0) = t_{mf}(1) = t_{mf}(3) = 2n - 1$; $t_{mf}(2) = 2n$.
□

Theorem 4.2. The graph $(P_n \odot K_1) \odot K_{1,n}$ is 4-total mean cordial for all values of $n \geq 2$.

Proof Let us now denote by $V((P_n \odot K_1) \odot K_{1,n}) = \{u_i, v_i, v_{i,j} : 1 \leq i \leq n, 1 \leq i \leq n\}$ and $E((P_n \odot K_1) \odot K_{1,n})=\{u_iv_i, v_iv_{i,j} : 1 \leq i \leq n, 1 \leq j \leq n\} \cup \{u_iu_{i+1} : 1 \leq i \leq n-1\}$ the vertex and the edge sets of $(P_n \odot K_1) \odot K_{1,n}$ respectively. Note that

$$|V((P_n \odot K_1) \odot K_{1,n})| + |E((P_n \odot K_1) \odot K_{1,n})| = 2n^2 + 4n - 1.$$

Case 1. $n \equiv 0 \pmod 4$.
Let $n = 4r$, $r \in \mathbb{N}$. Assign the label 0 to the r vertices $u_1, u_2, \ldots, u_r$. Now we assign the label 1 to the r vertices $u_{r+1}, u_{r+2}, \ldots, u_{2r}$. Next we assign the label 2 to the r vertices $u_{2r+1}, u_{2r+2}, \ldots, u_{3r}$. We now assign the label 3 to the r vertices $u_{3r+1}, u_{3r+2}, \ldots, u_{4r}$.
Assign the label 0 to the r vertices $v_1, v_2, \ldots, v_r$. Next we assign the label 1 to the r vertices $v_{r+1}, v_{r+2}, \ldots, v_{2r}$. Now we assign the label 2 to the r vertices $v_{2r+1}, v_{2r+2}, \ldots, v_{3r}$. Then we assign the label 3 to the r vertices $v_{3r+1}, v_{3r+2}, \ldots, v_{4r}$.
Assign the label 0 to the r copies of $4r$ vertices $v_{1,1}, v_{1,2}, \ldots, v_{1,4r}, v_{2,1}, v_{2,2}, \ldots, v_{2,4r}, \ldots, v_{r,1}, v_{r,2}, \ldots, v_{r,4r}$. We now assign the label 1 to the r copies of $4r$ vertices $v_{r+1,1}, v_{r+1,2}, \ldots, v_{r+1,4r}, v_{r+2,1}, v_{r+2,2}, \ldots, v_{r+2,4r}, \ldots, v_{2r,1}, v_{2r,2}, \ldots, v_{2r,4r}$. Now we assign the label 2 to the r copies of $4r$ vertices $v_{2r+1,1}, v_{2r+1,2}, \ldots, v_{2r+1,4r}, v_{2r+2,1}, v_{2r+2,2}, \ldots, v_{2r+2,4r}, \ldots, v_{3r,1}, v_{3r,2}, \ldots, v_{3r,4r}$. Finally we assign the label 3 to the r copies of $4r$ vertices $v_{3r+1,1}, v_{3r+1,2}, \ldots, v_{3r+1,4r}, v_{3r+2,1}, v_{3r+2,2}, \ldots, v_{3r+2,4r}, \ldots, v_{4r,1}, v_{4r,2}, \ldots, v_{4r,4r}$.

Case 2. $n \equiv 1 \pmod 4$.
Let $n = 4r + 1$, $r \in \mathbb{N}$. Label the vertices u_i, v_i, $v_{i.j}$ $(1 \leq i \leq 4r)$, $(1 \leq j \leq 4r)$ as in Case 1. We now assign the labels 2, 0 to the vertices u_{4r+1}, v_{4r+1}. Next we assign the label 0 to the $r+1$ vertices $v_{4r+1,1}, v_{4r+1,2}, \ldots, v_{4r+1,r+1}$. Now we assign the label 1 to the r

vertices $v_{4r+1,r+2}$, $v_{4r+1,r+3}$, $\ldots$, $v_{4r+1,2r+1}$. Finally we assign the label 3 to the $2r$ vertices $v_{4r+1,2r+2}$, $v_{4r+1,2r+3}$, $\ldots$, $v_{4r+1,4r+1}$.

Case 3. $n \equiv 2 \pmod 4$.
Let $n = 4r + 2$, $r \in \mathbb{N}$. As in Case 1, we now assign the label to the vertices u_i, v_i, $v_{i.j}$ $(1 \le i \le 4r, 1 \le j \le 4r)$. Now we assign the labels 3, 0, 0, 1 to the vertices u_{4r+1}, u_{4r+2}, v_{4r+1}, v_{4r+2}. Next we assign the label 3 to the $4r + 2$ vertices $v_{4r+1,1}$, $v_{4r+1,2}$, $\ldots$, $v_{4r+1,4r+2}$. Finally we assign the label 0 to the $4r+2$ vertices $v_{4r+2,1}$, $v_{4r+2,2}$, $\ldots$, $v_{4r+2,4r+2}$.

Case 4. $n \equiv 3 \pmod 4$.
Let $n = 4r + 3$, $r \in \mathbb{N}$. Label the vertices u_i, v_i, $v_{i.j}$ $(1 \le i \le 4r + 2)$, $(1 \le j \le 4r + 2)$ as in Case 3. Next we assign the labels 1, 0 to the vertices u_{4r+3}, v_{4r+3}. Now we assign the label 0 to the $r + 1$ vertices $v_{4r+3,1}$, $v_{4r+3,2}$, $\ldots$, $v_{4r+3,r+1}$. We now assign the label 1 to the r vertices $v_{4r+3,r+2}$, $v_{4r+3,r+3}$, $\ldots$, $v_{4r+3,2r+1}$. Finally we assign the label 3 to the $2r + 2$ vertices $v_{4r+3,2r+2}$, $v_{4r+3,2r+3}$, $\ldots$, $v_{4r+3,4r+3}$.
Thus this vertex labeling f is a 4-total mean cordial labeling follows from the Table 1 and Table 2.

n	$t_{mf}(0)$	$t_{mf}(1)$
$n = 4r$	$4r\,(2r + 1) - 1$	$4r\,(2r + 1)$
$n = 4r + 1$	$8r\,(r + 1) + 2$	$8r\,(r + 1) + 1$
$n = 4r + 2$	$4r\,(2r + 3) + 3$	$4r\,(2r + 3) + 4$
$n = 4r + 3$	$8r\,(r + 2) + 7$	$8r\,(r + 2) + 8$

Table 1: vertex labeling f (1)

n	$t_{mf}(2)$	$t_{mf}(3)$
$n = 4r$	$4r\,(2r + 1)$	$4r\,(2r + 1)$
$n = 4r + 1$	$8r\,(r + 1) + 1$	$8r\,(r + 1) + 1$
$n = 4r + 2$	$4r\,(2r + 3) + 4$	$4r\,(2r + 3) + 4$
$n = 4r + 3$	$8r\,(r + 2) + 7$	$8r\,(r + 2) + 7$

Table 2: vertex labeling f (2)

Case 5. $n = 2, 3$.
A 4-total mean cordial labeling of $(P_2 \odot K_1) \odot K_{1,2}$, $(P_3 \odot K_1) \odot K_{1,3}$ is given in Table 3 and Table 4.

n	u_1	u_2	u_3	v_1	v_2	v_3
2	2	3	–	0	1	–
3	3	3	3	0	1	1

Table 3: $(P_2 \odot K_1) \odot K_{1,2}$

n	$v_{1,1}$	$v_{1,2}$	$v_{1,3}$	$v_{2,1}$	$v_{2,2}$	$v_{2,3}$	$v_{3,1}$	$v_{3,2}$	$v_{3,3}$
2	3	3	–	0	0	–	–	–	–
3	0	0	0	1	1	1	2	3	3

Table 4: $(P_3 \odot K_1) \odot K_{1,3}$

□

Example 4.3. A 4 - total mean cordial labeling of $(P_5 \odot K_1) \odot K_{1,5}$ is given in figure 1.

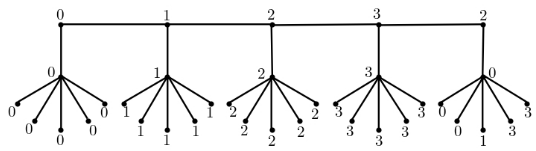

Figure 1: 4 - total mean cordial labeling of $(P_5 \odot K_1) \odot K_{1,5}$

Theorem 4.4. $G^{(n)}$ is 4-total mean cordial for all values of $n \geq 2$.

Proof Take the vertex set and the edge set of $G^{(n)}$ as in Definition 3.5.
Note that $\left|V\left(G_{(n)}\right)\right| + \left|E\left(G_{(n)}\right)\right| = 2n^2 + 4n - 1$.

Case 1. $n \equiv 0 \pmod 4$.
Let $n = 4r$, $r \in \mathbb{N}$. Assign the label 0 to the r vertices $u_1, u_2, \ldots, u_r$. Now we assign the label 1 to the r vertices $u_{r+1}, u_{r+2}, \ldots, u_{2r}$. Next we assign the label 2 to the r vertices $u_{2r+1}, u_{2r+2}, \ldots, u_{3r}$. Then we assign the label 3 to the r vertices $u_{3r+1}, u_{3r+2}, \ldots, u_{4r}$.
Now we assign the label 0 to the r vertices $v_1, v_2, \ldots, v_r$. Next we assign the label 1 to the r vertices $v_{r+1}, v_{r+2}, \ldots, v_{2r}$. We now assign the label 2 to the r vertices $v_{2r+1}, v_{2r+2}, \ldots, v_{3r}$. Next we assign the label 3 to the r vertices $v_{3r+1}, v_{3r+2}, \ldots, v_{4r}$.
Next we assign the label 0 to the r vertices $w_1, w_2, \ldots, w_r$. Now we assign the label 1 to the r vertices $w_{r+1}, w_{r+2}, \ldots, w_{2r}$. We now assign the label 2 to the r vertices $w_{2r+1}, w_{2r+2}, \ldots, w_{3r}$. Next we assign the label 3 to the r vertices $w_{3r+1}, w_{3r+2}, \ldots, w_{4r}$.
Assign the label 0 to the r copies of $4r-1$ vertices $w_{1,1}, w_{1,2}, \ldots, w_{1,4r-1}, w_{2,1}, w_{2,2}, \ldots, w_{2,4r-1}, \ldots, w_{r,1}, w_{r,2}, \ldots, w_{r,4r-1}$. Next we assign the label 1 to the r copies of $4r-1$ vertices $w_{r+1,1}, w_{r+1,2}, \ldots, w_{r+1,4r-1}, w_{r+2,1}, w_{r+2,2}, \ldots, w_{r+2,4r-1}, \ldots, w_{2r,1}, w_{2r,2}, \ldots,$

$w_{2r,4r-1}$. Now we assign the label 2 to the r copies of $4r-1$ vertices $w_{2r+1,1}$, $w_{2r+1,2}$, $\ldots$, $w_{2r+1,4r-1}$,$w_{2r+2,1}$, $w_{2r+2,2}$, $\ldots$, $w_{2r+2,4r-1}$, $\ldots$, $w_{3r,1}$, $w_{3r,2}$, $\ldots$, $w_{3r,4r-1}$. Finally we assign the label 3 to the r copies of $4r-1$ vertices $w_{3r+1,1}$, $w_{3r+1,2}$, $\ldots$, $w_{3r+1,4r-1}$,$w_{3r+2,1}$, $w_{3r+2,2}$, $\ldots$, $w_{3r+2,4r-1}$, $\ldots$, $w_{4r,1}$, $w_{4r,2}$, $\ldots$, $w_{4r,4r-1}$.

Case 2. $n \equiv 1 \pmod 4$.
Let $n = 4r+1$, $r \in \mathbb{N}$. As in case 1, Label the vertices u_i, v_i, w_i, $w_{i.j}$ $(1 \leq i \leq 4r)$, $(1 \leq j \leq 4r-1)$. Now we assign the labels 3, 1, 0 to the vertices u_{4r+1}, v_{4r+1}, w_{4r+1}. Next we assign the label 0 to the $r+1$ vertices $w_{4r+1,1}$, $w_{4r+1,2}$, $\ldots$, $w_{4r+1,r+1}$. We now assign the label 1 to the $r-1$ vertices $w_{4r+1,r+2}$, $w_{4r+1,r+3}$, $\ldots$, $w_{4r+1,2r}$. Now we assign the label 2 to the vertex $w_{4r+1,2r+1}$. Finally we assign the label 3 to the $2r-1$ vertices $w_{4r+1,2r+2}$, $w_{4r+1,2r+3}$, $\ldots$, $w_{4r+1,4r}$.

Case 3. $n \equiv 2 \pmod 4$.
Let $n = 4r+2$, $r \in \mathbb{N}$. Now we assign the label to the vertices u_i, v_i, w_i, $w_{i.j}$ $(1 \leq i \leq 4r)$, $(1 \leq j \leq 4r-1)$, as in case 1. we now assign the labels 2, 3, 0, 0, 0, 1 to the vertices u_{4r+1}, u_{4r+2}, v_{4r+1}, v_{4r+2}, w_{4r+1}, w_{4r+2}. Next we assign the label 3 to the $4r+1$ vertices $w_{4r+1,1}$, $w_{4r+1,2}$, $\ldots$, $w_{4r+1,4r+1}$. Finally we assign the label 0 to the $4r+1$ vertices $w_{4r+2,1}$, $w_{4r+2,2}$, $\ldots$, $w_{4r+2,4r+1}$.

Case 4. $n \equiv 3 \pmod 4$.
Let $n = 4r+3$, $r \in \mathbb{N}$. Label the vertices u_i, v_i, w_i, $w_{i.j}$ $(1 \leq i \leq 4r+2)$, $(1 \leq j \leq 4r+1)$ as in Case 3. Next we assign the labels 2, 1, 0 to the vertices u_{4r+3}, v_{4r+3}, w_{4r+3}. Now we assign the label 0 to the $r+1$ vertices $w_{4r+3,1}$, $w_{4r+3,2}$, $\ldots$, $w_{4r+3,r+1}$. Next we assign the label 1 to the r vertices $w_{4r+3,r+2}$, $w_{4r+3,r+3}$, $\ldots$, $w_{4r+3,2r+1}$. Finally we assign the label 3 to the $2r+1$ vertices $w_{4r+3,2r+2}$, $w_{4r+3,2r+3}$, $\ldots$, $w_{4r+3,4r+2}$.
f is a 4-total mean cordial labeling follows from the Table 5 and Table 6.

order of n	$t_{mf}(0)$	$t_{mf}(1)$
$n = 4r$	$4r\,(2r+1)-1$	$4r\,(2r+1)$
$n = 4r+1$	$8r\,(r+1)+2$	$8r\,(r+1)+1$
$n = 4r+2$	$4r\,(2r+3)+4$	$4r\,(2r+3)+4$
$n = 4r+3$	$8r\,(r+2)+8$	$8r\,(r+2)+7$

Table 5: $t_{mf}(0,1)$

order of n	$t_{mf}(2)$	$t_{mf}(3)$
$n = 4r$	$4r\,(2r+1)$	$4r\,(2r+1)$
$n = 4r+1$	$8r\,(r+1)+1$	$8r\,(r+1)+1$
$n = 4r+2$	$4r\,(2r+3)+3$	$4r\,(2r+3)+4$
$n = 4r+3$	$8r\,(r+2)+7$	$8r\,(r+2)+7$

Table 6: $t_{mf}(2,3)$

Case 5. $n = 2, 3$.
4 - total mean cordial labeling of $G^{(2)}$, $G^{(3)}$ is given in Table 7 and Table 8.

n	u_1	u_2	u_3	v_1	v_2	v_3	w_1	w_2	w_3
2	0	0	–	1	1	–	2	2	–
3	2	3	2	0	0	1	0	1	0

Table 7: 4 - total mean cordial labeling of $G^{(2)}$, $G^{(3)}$

n	$w_{1,1}$	$w_{1,2}$	$w_{2,1}$	$w_{2,2}$	$w_{3,1}$	$w_{3,2}$
2	3	3	–	–	–	–
3	3	3	0	0	3	3

Table 8: 4 - total mean cordial labeling of $G^{(2)}$, $G^{(3)}$

□

Theorem 4.5. $BT\,(n,n)$ is 4-total mean cordial for all values of $n \geq 2$.

Proof Let us now denote by $V\,(BT\,(n,n)) = \{u, u_i, v_i, v_{i,j} : 1 \leq i \leq n, 1 \leq j \leq n-2\}$ and $E\,(BT\,(n,n))$=$\{uu_i, u_iv_i, v_iv_{i,j} : 1 \leq i \leq n, 1 \leq j \leq n-2\}$ respectively the vertex and edge sets of $BT\,(n,n)$. Clearly $|V\,(BT\,(n,n))| + |E\,(BT\,(n,n))| = 2n^2 + 1$.

Assign the label 3 to the vertex u. Now we assign the label 2 to the n vertices u_1, u_2, …, u_n. Next we assign the label 0 to the n vertices v_1, v_2, …, v_n.

Case 1. $n \equiv 0 \pmod 4$.
Let $n = 4r$, $r \in \mathbb{N}$. Assign the label 0 to the r copies of $4r-2$ vertices $v_{1,1}$, $v_{1,2}$, …, $v_{1,4r-2}$,$v_{2,1}$, $v_{2,2}$, …, $v_{2,4r-2}$, …, $v_{r,1}$, $v_{r,2}$, …, $v_{r,4r-2}$. Now we assign the label 1 to the r copies of $4r-2$ vertices $v_{r+1,1}$, $v_{r+1,2}$, …, $v_{r+1,4r-2}$,$v_{r+2,1}$, $v_{r+2,2}$, …, $v_{r+2,4r-2}$, …, $v_{2r,1}$, $v_{2r,2}$, …, $v_{2r,4r-2}$. Next we assign the label 3 to the $2r$ copies of $4r-2$ vertices $v_{2r+1,1}$, $v_{2r+1,2}$, …, $v_{2r+1,4r-2}$,$v_{2r+2,1}$, $v_{2r+2,2}$, …, $v_{2r+2,4r-2}$, …, $w_{3r,1}$, $w_{3r,2}$, …, $w_{3r,4r-2}$ $v_{3r+1,1}$, $v_{3r+1,2}$, …, $v_{3r+1,4r-2}$,$v_{3r+2,1}$, $v_{3r+2,2}$, …, $v_{3r+2,4r-2}$, …, $v_{4r,1}$, $v_{4r,2}$, …, $v_{4r,4r-2}$.

Case 2. $n \equiv 1 \pmod 4$.
Let $n = 4r+1$, $r \in \mathbb{N}$. As in case 1, Label the vertices $v_{i,j}$ $(1 \le i \le 4r)$, $(1 \le j \le 4r-2)$. Next we assign the label 0 to the r vertices $v_{4r+1,1}$, $v_{4r+1,2}$, $\ldots$, $v_{4r+1,r}$. We now assign the label 1 to the r vertices $v_{4r+1,r+1}$, $v_{4r+1,r+2}$, $\ldots$, $v_{4r+1,2r}$. Finally we assign the label 3 to the $2r-1$ vertices $v_{4r+1,2r+1}$, $v_{4r+1,2r+2}$, $\ldots$, $v_{4r+1,4r-1}$.

Case 3. $n \equiv 2 \pmod 4$.
Let $n = 4r+2$, $r \ge 0$. Now we assign the label to the vertices $v_{i,j}$ $(1 \le i \le 4r)$, $(1 \le j \le 4r-2)$, as in case 1. We now assign the label 0 to the r vertices $v_{4r+1,1}$, $v_{4r+1,2}$, $\ldots$, $v_{4r+1,r}$. Next we assign the label 1 to the r vertices $v_{4r+1,r+1}$, $v_{4r+1,r+2}$, $\ldots$, $v_{4r+1,2r}$. Then we assign the label 3 to the $2r$ vertices $v_{4r+1,2r+1}$, $v_{4r+1,2r+2}$, $\ldots$, $v_{4r+1,4r}$.
We now assign the label 0 to the r vertices $v_{4r+2,1}$, $v_{4r+2,2}$, $\ldots$, $v_{4r+2,r}$. Next we assign the label 1 to the r vertices $v_{4r+2,r+1}$, $v_{4r+2,r+2}$, $\ldots$, $v_{4r+2,2r}$. Finally we assign the label 3 to the $2r$ vertices $v_{4r+2,2r+1}$, $v_{4r+2,2r+2}$, $\ldots$, $v_{4r+2,4r}$.

Case 4. $n \equiv 3 \pmod 4$.
Let $n = 4r+3$, $r \ge 0$. Label the vertices $v_{i,j}$ $(1 \le i \le 4r)$, $(1 \le j \le 4r-2)$, as in case 1. Now we assign the label 0 to the $r+1$ vertices $v_{4r+1,1}$, $v_{4r+1,2}$, $\ldots$, $v_{4r+1,r+1}$. Next we assign the label 1 to the r vertices $v_{4r+1,r+2}$, $v_{4r+1,r+3}$, $\ldots$, $v_{4r+1,2r+1}$. Then we assign the label 3 to the $2r$ vertices $v_{4r+1,2r+2}$, $v_{4r+1,2r+3}$, $\ldots$, $v_{4r+1,4r+1}$.
Next we assign the label 0 to the r vertices $v_{4r+2,1}$, $v_{4r+2,2}$, $\ldots$, $v_{4r+2,r}$. Now we assign the label 1 to the $r+1$ vertices $v_{4r+2,r+1}$, $v_{4r+2,r+2}$, $\ldots$, $v_{4r+2,2r+1}$. We now assign the label 3 to the $2r$ vertices $v_{4r+2,2r+2}$, $v_{4r+2,2r+3}$, $\ldots$, $v_{4r+2,4r+1}$.
Now we assign the label 0 to the r vertices $v_{4r+3,1}$, $v_{4r+3,2}$, $\ldots$, $v_{4r+3,r}$. Next we assign the label 1 to the r vertices $v_{4r+3,r+1}$, $v_{4r+3,r+2}$, $\ldots$, $v_{4r+3,2r}$. Finally we assign the label 3 to the $2r+1$ vertices $v_{4r+3,2r+1}$, $v_{4r+3,2r+2}$, $\ldots$, $v_{4r+3,4r+1}$.

Thus this vertex labeling f is a 4-total mean cordial labeling follows from the Table 9 and Table 10.

size of n	$t_{mf}(0)$	$t_{mf}(1)$
$n = 4r$	$8r^2$	$8r^2$
$n = 4r+1$	$4r\,(2r+1)+1$	$4r\,(2r+1)+1$
$n = 4r+2$	$8r\,(r+1)+2$	$8r\,(r+1)+2$
$n = 4r+3$	$4r\,(2r+3)+5$	$4r\,(2r+3)+5$

Table 9: $t_{mf}(0,1)$

□

size of n	$t_{mf}(2)$	$t_{mf}(3)$
$n = 4r$	$8r^2$	$8r^2 + 1$
$n = 4r + 1$	$4r\,(2r + 1)$	$4r\,(2r + 1) + 1$
$n = 4r + 2$	$8r\,(r + 1) + 2$	$8r\,(r + 1) + 3$
$n = 4r + 3$	$4r\,(2r + 3) + 4$	$4r\,(2r + 3) + 5$

Table 10: $t_{mf}(2, 3)$

Example 4.6. A 4 - total mean cordial labeling of $BT(4, 4)$ is given in figure 2.

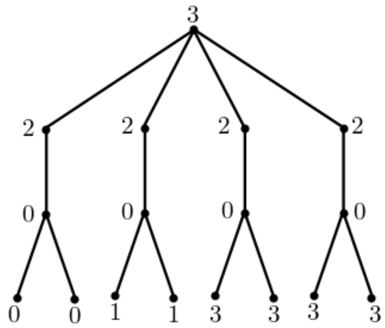

Figure 2: 4 - total mean cordial labeling of $BT(4, 4)$

Theorem 4.7. The coconut tree $CT\,(n, n)$ is 4-total mean cordial for all values of $n \geq 2$.

Proof Let P_n be the path $u_1\ u_2\ \ldots\ u_n$ and $V\,(K\,(1, n)) =$ $\{v, v_i : 1 \leq i \leq n\ \ and\ \ v = u_n\}$ are the vertex set of $CT\,(n, n)$ and edge set $E\,(CT\,(n, n))$=$E\,(P_n) \cup$ $E\,(K_{1,n})$.
Obviously $|V\,(CT\,(n, n))| + |E\,(CT\,(n, n))| = 4n - 1$.

Case 1. $n \equiv 0 \pmod 2$.
Let $n = 2r$, $r \in \mathbb{N}$. Assign the label 0 to the r vertices u_1, u_2, $\ldots$, u_r. Now we assign the label 1 to the r vertices u_{r+1}, u_{r+2}, $\ldots$, u_{2r}. Next move to the vertices of the star $K_{1,n}$. We now assign the label 3 to the $2r$ vertices v_1, v_2, $\ldots$, v_{2r}.

Case 2. $n \equiv 1 \pmod 2 r$.
Let $n = 2r + 1$, $r \in \mathbb{N}$. Assign the label 0 to the $r + 1$ vertices u_1, u_2, $\ldots$, u_{r+1}. Next we assign the label 1 to the r vertices u_{r+2}, u_{r+3}, $\ldots$, u_{2r+1}. Finally we assign the label 3 to the $2r + 1$ vertices v_1, v_2, $\ldots$, v_{2r+1}.

f is a 4-total mean cordial labeling follows from the Table 11.

n	$t_{mf}(0)$	$t_{mf}(1)$	$t_{mf}(2)$	$t_{mf}(3)$
$n = 2r$	$2r - 1$	$2r$	$2r$	$2r$
$n = 2r + 1$	$2r + 1$	$2r$	$2r + 1$	$2r + 1$

Table 11: $t_{mf}(0, 1, 2, 3)$

□

Example 4.8. A 4 - total mean cordial labeling of $CT(5, 5)$ is given in figure 3.

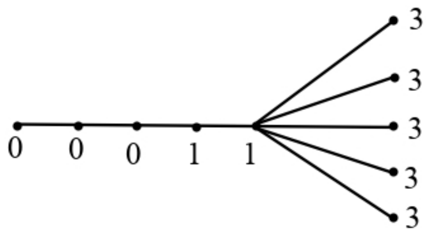

Figure 3: 4 - total mean cordial labeling of $CT(5, 5)$

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