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The Stationary-NonStationary Process and The Variable Roots' Difference Equations

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Abstract
Stochastic, processes can be stationary or nonstationary. They depend on the magnitude of shocks. In other words, in an auto regressive model of order one, the estimated coefficient is not constant. Another finding of this paper is the relation between estimated coefficients and residuals. We also develop a catastrophe and chaos theory for change of roots from stationary to a nonstationary one and vice versa.
Key words: Stationarity, multi equilibrium, catastrophe points

1- Introduction

The equilibrium concept is a frequently used but not necessarily well understood. In physics, three kinds of equilibria points are distinguished from each other: stable, unstable and neutral. A stable equilibrium, when shocked, deviates from its initial position, then finally reverts back to its first point.

This condition is not full filled for unstable equilibria. The neutral equilibrium is not very sensitive to the equilibrium point. Fig.1 shows these concepts clearly.

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Table 1

Catastrophe	Control Dimensions	Behavior dimension	Function	First derivative
Fold	1	1	$\frac{1}{3}z^3 - XZ$	$Z^2 - X$
Cusp	2	1	$\frac{1}{4}z^4 - XZ - \frac{1}{2}z^2Y$	$Z^3 - X - Yz$
Swallowation	3	1	$\frac{1}{5}z^5 - XZ - \frac{1}{2}Yz^2 - \frac{1}{3}Vz^3$	$Z^4 - X - YZ - Vz^2$
Butterfly	4	1	$\frac{1}{6}z^6 - XZ - \frac{1}{2}Yz^2 - \frac{1}{3}Vz^3 - \frac{1}{4}Uz^4$	$Z^2 - X - YZ - VZ^2 - VZ^3$

Sources : Lange, R etal (2000)

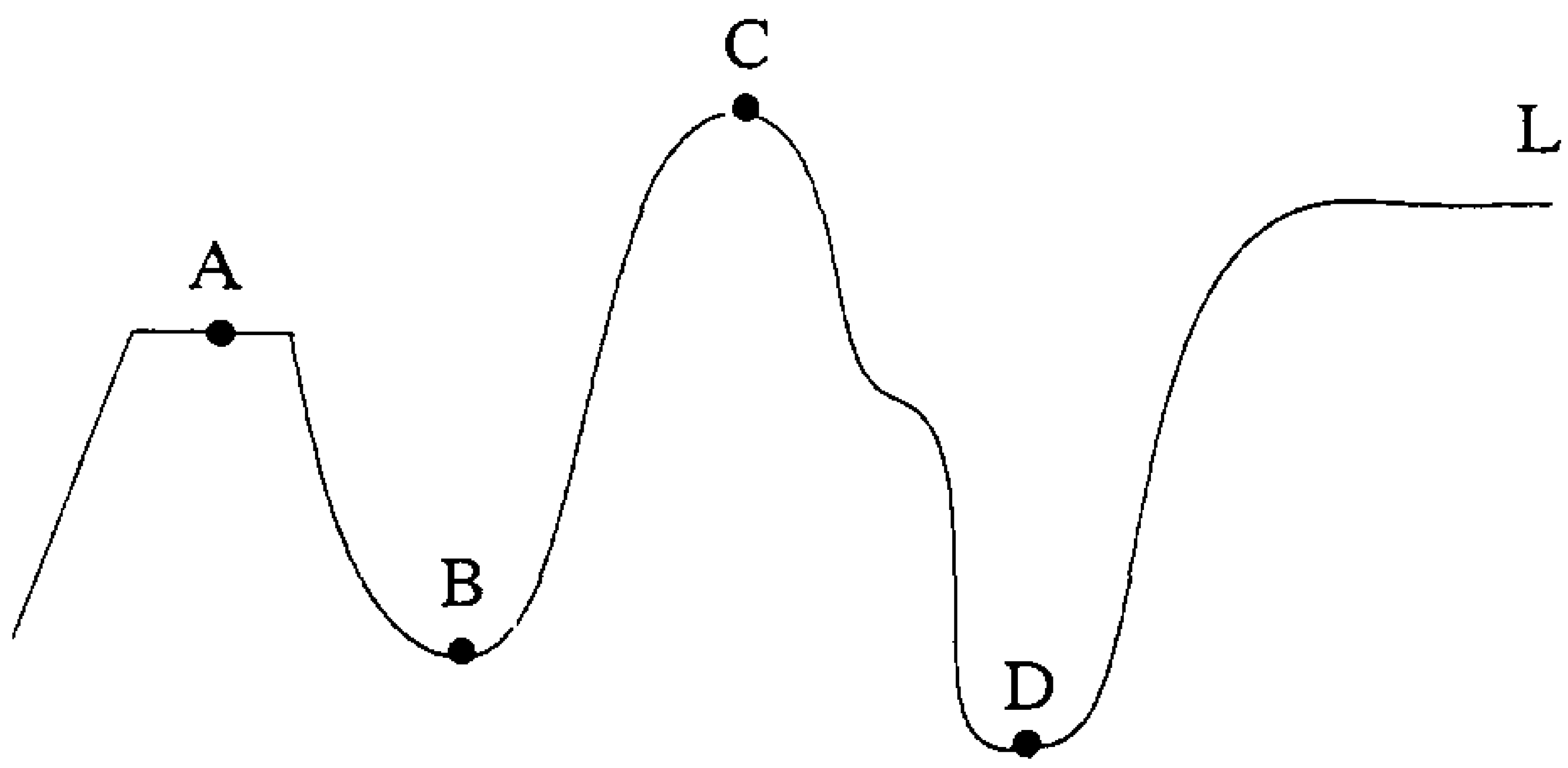


Fig. 1

"A" is a neutral equilibrium, "B" is stable and "C" is an unstable equilibrium point. If a ball falls on "B" and a weak shock hits it, it will stop on "B" after some oscillations. However, if a very strong shock hits the ball, it is possible that the ball will go up and stay on "C" point which is an unstable point. Therefore, the magnitude of shock is important in the determination of the system stability and equilibrium point. It is better to say that "B" is a stable point but for shocks with specific magnitude, say δ, instead of saying that "B" is stable (absolutely). Also "C" can be stable but for very small shocks of, for example, ε magnitude. It is easy to understand that in the real world, surfaces are not perfectly smooth and each interval has many microscopic "peaks" and "through" that can be stable and unstable points for very small shocks (ε-shocks). The counterpart of this specification is "variable root" differential equations "and in continious form" "variable root differential equations". In this equation:

$$y' + at + c = 0$$
$$\text{then} \quad y = ce^{-at} \tag{1}$$

The root "a" is not supposed to be nonconstant and may be the function of the magnitude of shocks in a stochastic differential equation. Therefore equation(1) can be stable or not depending on the magnitude of shock that is imposed on the model.

The rest of the paper is organized as follows: The next section debates about "Variable Root Differential Equations" (VRDE, henceforth) and stochastic unit root processes. In section III, we will cover the estimation of specific kinds of VRDE models and their applications to the stock market index. Further debates and generalizations are discussed in the fourth section. The final section is a conclusion and propositions for further studies.

2- Variable Root Differential Equations (VRDE)

In mathematical textbooks differential equations are distinguished by order, degree, ordinarily and partiality, and individuality or simultaneity. Some variable coefficient equations can be variable root equations that are stable for some conditions and unstable for other conditions. This concept is applicable for difference equations. Let a difference equation be as follows:

$$y_{t+1} = \alpha(z)y_t \tag{2}$$

α is function of z, and a variation in z will change α. Therefore :

$$y = A[\alpha(Z)]^t$$

$$\text{when } t \to \infty, y_t \to \begin{cases} \text{cons tan yif } |\alpha(z)| \le 1 \\ \text{cons tan yif } |\alpha(z)| \ge 1 \end{cases}$$

But $\alpha(z)$ is a function and can be less than one, or more than itself. “z” is proxy for every variable such as shocks in stochastic difference equations or another variable depending on the model builder's view.

A general form is this form:

$$y_{t+1} = \alpha_i(z) y_t$$

Such that I is an index for functional form and may be a function of other variables than z or even z's growth, difference and so on. Variation in values that are considered for I will change the functional form from exponential to a trigonometric one, for example. In this case there are two sources for variability of root. In this equation:

$$y_{t+1} = \alpha_i(\varepsilon_t) y_t + \varepsilon_{t+1}$$

or

$$y_{t+1} = \alpha_i(\varepsilon_{t-i}) y_t + \varepsilon_{t+1}$$

or

$$y_{t+1} = \alpha_i(\bar{\varepsilon}) y_t + \varepsilon_{t+1}$$

ε_t is an error term that is distributed normally with zero mean and variance of σ_ε^2. Regardless of solution of equations estimation, the process is not different from $y_{t+1} = \rho y_t + \varepsilon_{t-1}$. However, we will discuss in detail the estimations of these models and consequences of ignoring the relation between σ and ε. It is possible that ρ be the function of the number of observations.

If one can show this hypothesis, then a unit root test should be interpreted with more caution than before. It is worthy to note that because of the possibility of linearization with the Taylor series, all of the following forms will be delivered to a linear one.

$$y_t = \rho(\varepsilon_t) y_{t-1} + \varepsilon_t$$

$$y_t = \rho \mathrm{Sign}(\varepsilon_t) y_{t-1} + \varepsilon_t$$

$$y_t = \rho(\varepsilon_t) y_{t-1} + \varepsilon_t$$

or

$$y_t = \rho(n) y_{t-1} + \varepsilon_t$$

In the recent equation, "n" is sample size. In this paper, we tend to verify these hyphotheses:

H_0 : $\hat{\rho}$ is not constant for various samples.

H_a : $\hat{\rho}$ is constant for various samples.

and

H_0 : $\hat{\rho}$ is a function of ε_t

H_a : $\hat{\rho}$ is not a function of ε_t

We will come back to the procedure of testing in the next section. Let's point to some literature about the nonconstancy of ρ in the nonstationary process. Granger and Swanson (1997) showed that there are some processes that have stochastic unit roots. They are stationary for some periods and mildly explosive for others. They give some insights about the power of the DF+ADF test against the stochastic unit root process.

The stochastic unit root processes have been studied by Osborn (1988), Franses (1994), Brandt(1986) and Pourahmadi(1986). In Göke (1999), the stability and existence of equilibrium is considered in catastrophe and hysteresis debates. In this paper, sudden jumps are named catastrophe. Fig. 2 shows different kinds of equilibra situations.

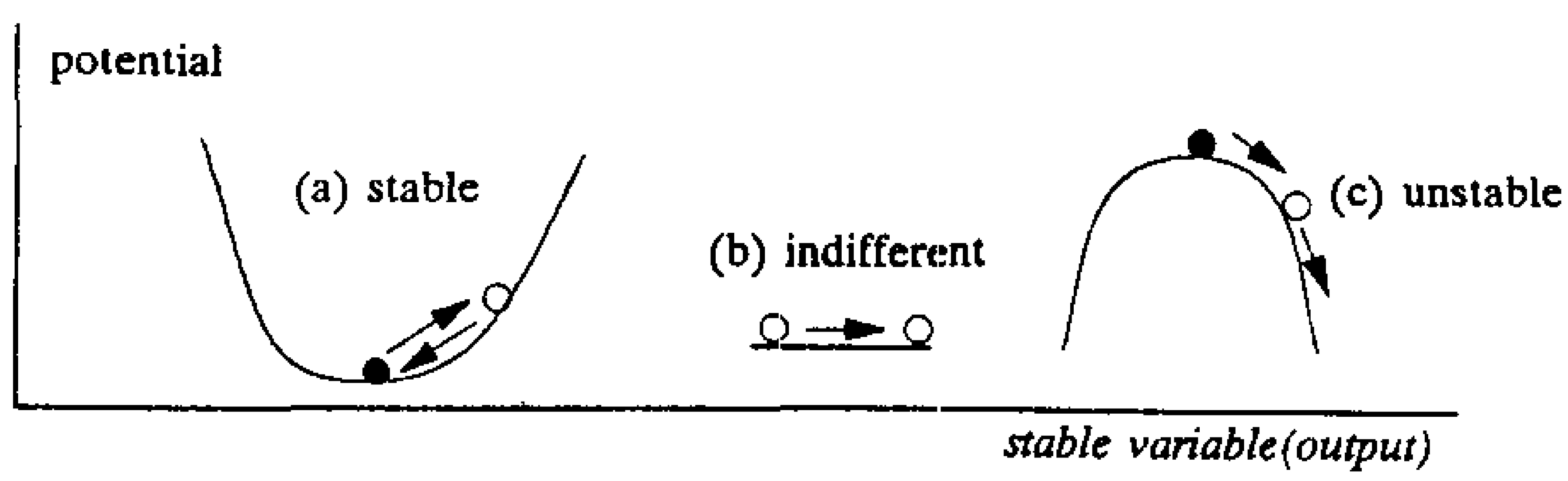


Fig. 2

source : Göcke (1999)

The graph above is for the potential function of second degree. The Cusp models are used in Göke (1999) for the analysis of economic hysteresis. Here, we will use some counterparts of the mentioned models for clarifying theoretical aspects of our work.

In Granger and Swanson (1997) a variable root difference equation (although not labeled VRDE) was used with the following form:

$$X_t = a_t X_t + \varepsilon_t$$

$$\varepsilon_t \sim \text{i.i.d}(0, \sigma_\varepsilon^2 \text{ and } a_t = \exp(\alpha_t)$$

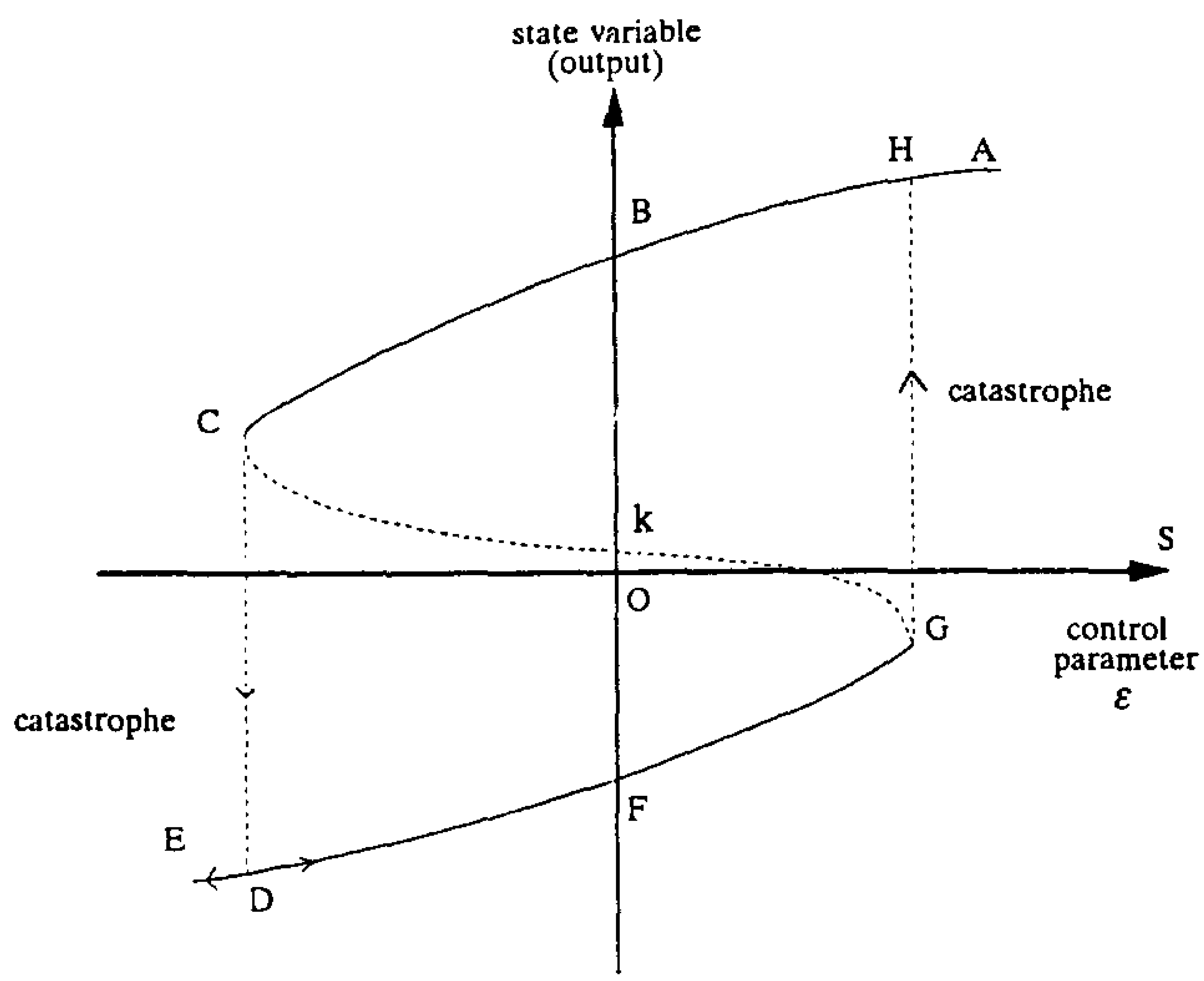
Where α_t , a Gussian stationary series, has mean m, variance σ^2_α and power spectrum $g_\alpha(w)$. Also α_t is generated exogenously from x_t , so that:

$$g(\alpha t+1 | \alpha_{t-j}, x_{t-j}, j\Re 0) = g(\alpha_{t+1} | \alpha_{t-j}, j\Re 0)$$

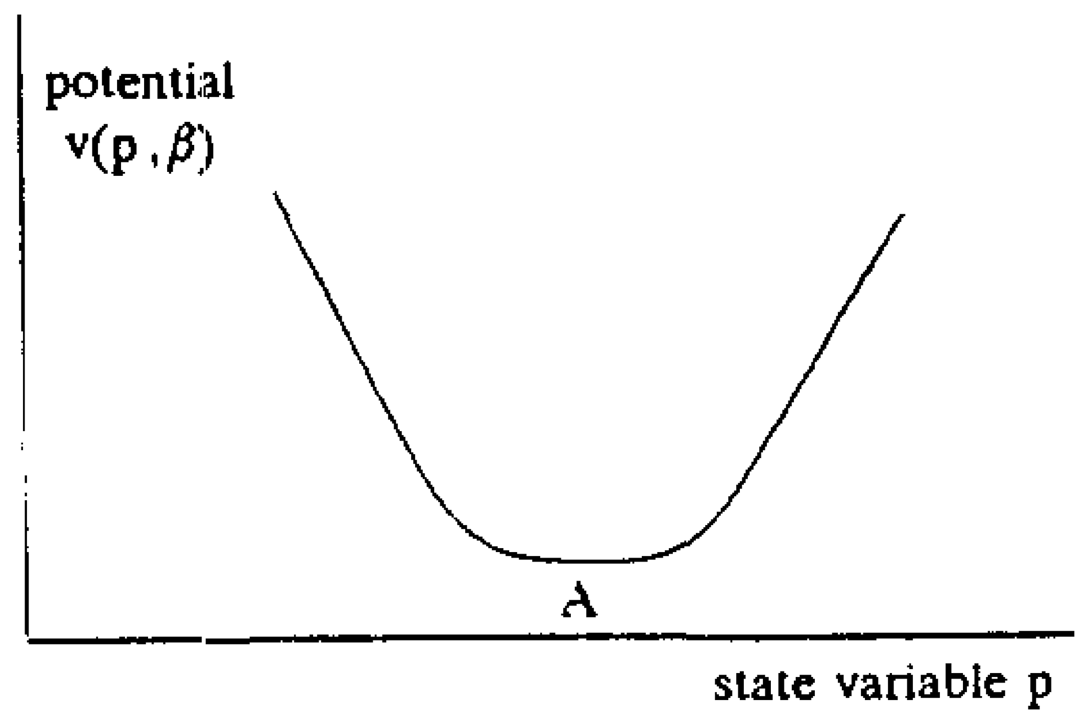
In this paper, we will not use this model and we don't tend to prove the possibility of raising the stochastic unit root. However we will try to show that for some processes such as the Tehran Stock Exchange, the index total ρ is not constant in the AR(1) model. As noted in the previous sector, ρ can be a function of ε_t(shocks), sample size or another variable. The consequences of this kind of dependence will be relativity in validity of unit root tests. It should be noted that we will focus on the violation of exogenity of ρ_i in respect to explanatory variables in section IV.

The second line of work is the determinants variability of ρ_i. In this way, we will consider the relation between ρ_i , ε_t and the variance of ε_t. It is important to know that the variations of ρ_i are smooth or sudden. It can be stated with the catastrophe theory. In Fig.2, potential functions of degree two were depicted. These kinds of equations have one kind of equilibrium. There are Cusp models with potential functions of degree four. This function is showed in Fig.3.

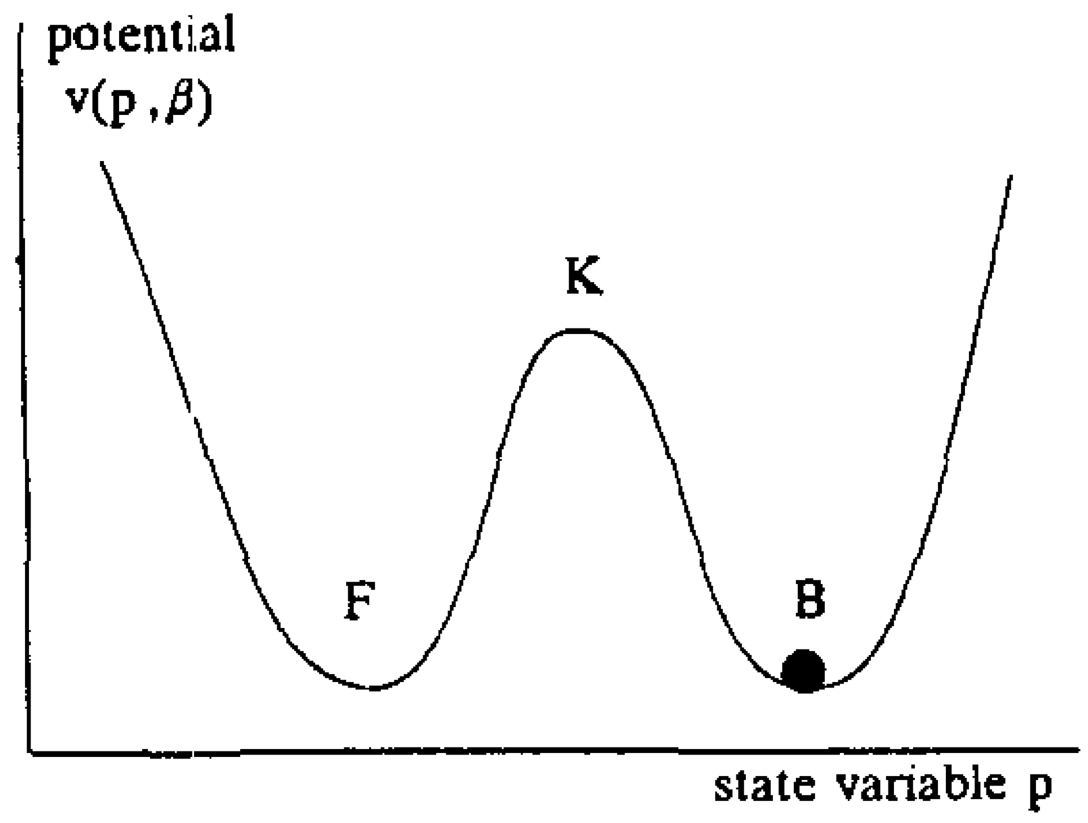
(a)



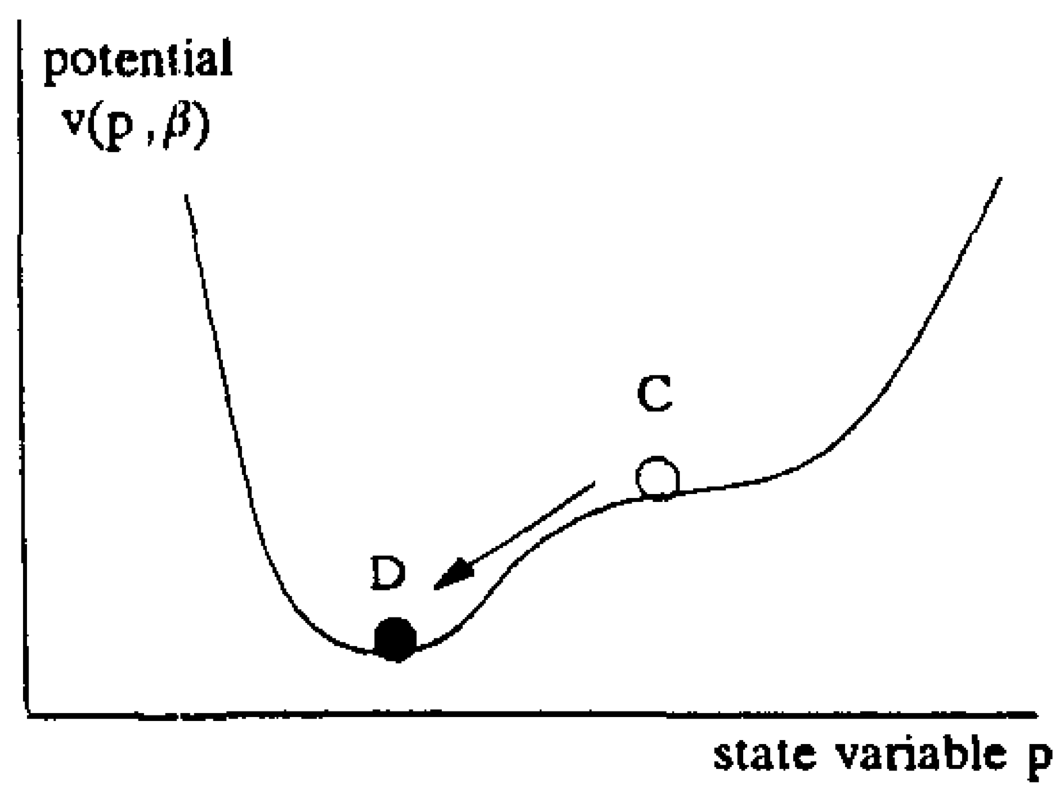
(b)



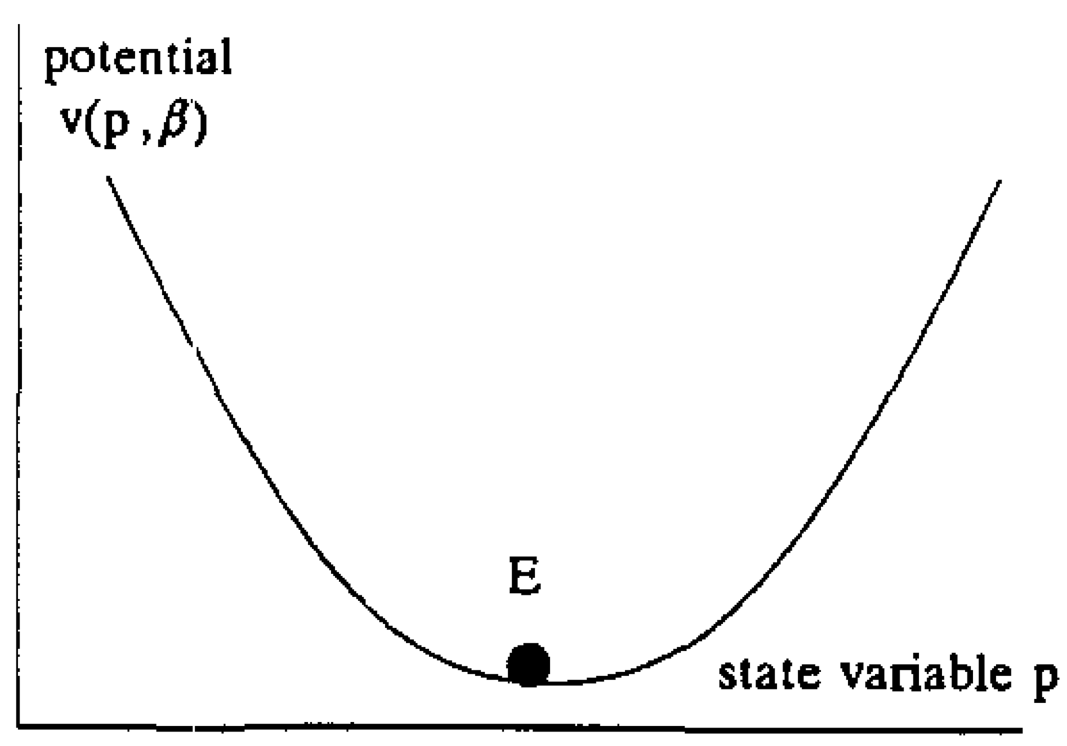
(c)



(d)



(e)



source: zeeman 1976, P68 and schuster 1991, P. 40

In part (a) of Fig.3, AHBC and EDFG curves depict the local minimum of the potential function (i:e. stable equilibria) and the dotted line CKG represents the local maximum (i.e. unstable equilibria). When ε (control variable) increases beyond s, the system jumps from "G" to "H". "H" is a stable equilibrium but there is no guarantee for achievement of "H" and it is possible that the system goes to "k" which is an unstable equilibrium. General models of the catastrophe theory can be summarized by the following Table 2.

Z is a dependent variable (state variable), X, Y, U, V are independent variables (control variables). All of the variables are latent (composite) variables:

$$X = \theta_{X^0} + \sum_{I=1}^{I} \theta_{X_i} X_i \quad , \quad Y = \theta_{Y^0} + \sum_{J=1}^{J} \theta_Y \quad , \quad Z = \theta_{Z^0} + \sum_{K=1}^{K} \theta_{ZK} Z_K$$

$$V = \theta_{V^0} + \sum_{S=1}^{S} \theta_{S_S} V_s \quad , \quad U = \theta_{U^0} + \sum_{P=1}^{P} \theta_{U_P} U_P$$

Only four forms of Thom's elementary catastrophes are presented in Table1. The three remaining forms are "Hyperbolic Ambilic" , " Eliptic Ambilic", and "Parabolic Ambilic". However we will not pay attention to these forms.

3- Empirical Results

For the test of the variability of ρ, we chose the Total Index of the Tehran Stock Exchange (TITSE) through the last eight years (i.e. February 1993 to February 2001). The observations are daily and they amounted to 1942 observations after the adjustments for the holiday. We regressed TITSE on TITSE(-1) and C for the 1942 observations and got ρ_1, The second time, we ran a regression with 1921 observations and got another value for ρ, say ρ_2. In later tests, we continued the process by reducing 30 days from the observation interval and finally got an 80 value for ρ (i.e. 80 distinct regressions from the sample size viewpoint).

These values are presented in (8-A). A test with statistic

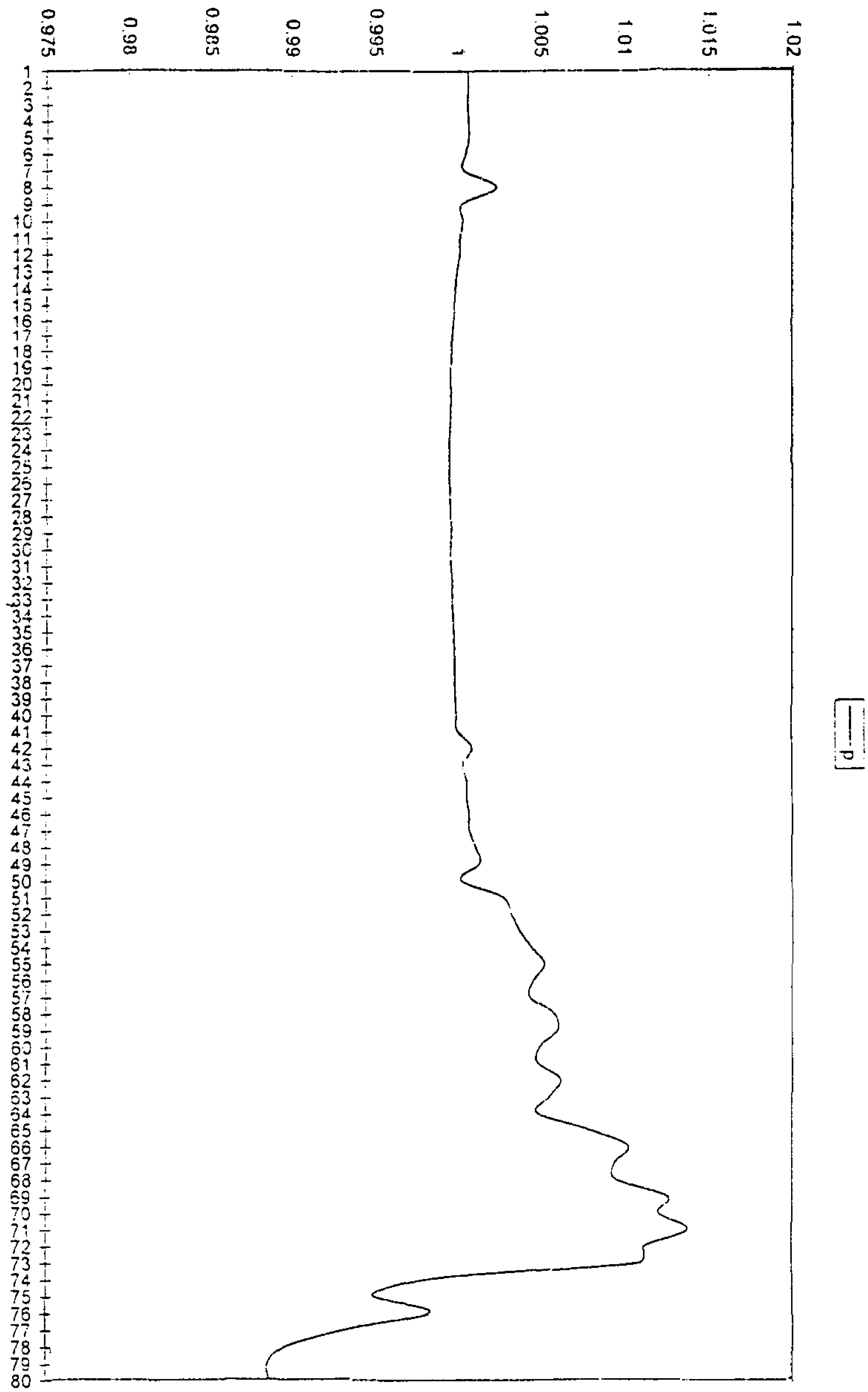
$$t = \frac{\rho\max - \rho\min}{SE(\rho)} = \frac{1.013837 - 0.988508}{0.004649} = 5.448298$$

shows that the difference between the max and min of ρ is significantly different from zero at common levels of significance. In other words, the volatility of ρ is important for statistical inferences. Consequences of this property are a greater doubt on unit root test's that rely on ρ_s estimates in AR(1) processes. However a further step is asking this question: What are the reasons for the unstability of ρ in various sample sizes? In order to answer it, we will focus on the relation between the mean of the absolute residuals(M) from each of the regressions and ρ_{is} that are obtained from corresponding regressions(8-A). These results show that there is a significant relation between the residuals and ρ_s as various samples. In general, the volatility of ρ can come from:

Mean, Standard Deviation and AR(1) Coefficients

	M		STDV	P	Accrual
M1	3.46616557	STA1	5.892624007	1.000568	238.6914448
M2	3.45601845	STA2	5.889794114	1.00058	237.1283431
M3	3.430715002	STA3	5.873005902	1.000534	242.8878728
M4	3.418352376	STA4	5.879185629	1.00061	233.6634859
M5	3.41287592	STA5	5.885620629	1.000623	232.1420981
M6	3.379772824	STA6	5.866351658	1.000417	256.4185369
M7	3.359923717	STA7	5.866056116	1.000322	267.7732085
M8	3.326364229	STA8	5.844324982	1.00231	278.1529787
M9	3.274114302	STA9	5.775927719	1.000233	277.9328881
M10	3.255491338	STA10	5.774042501	1.000272	273.6071502
M11	3.240543434	STA11	5.771933699	1.00017	284.4435843
M12	3.242230.479	STA12	5.789319166	1.000137	287.7729593
M13	3.201266109	STA13	5.748186004	0.999977	303.0778671
M14	3.18668766	STA14	5.757736558	0.99986	312.9996928
M15	3.19340749	STA15	5.783935583	0.99981	316.781317
M16	3.196416027	STA16	5.808219603	0.999763	319.9758064
M17	3.16808007	STA17	5.80461185	0.999669	325.0946591
M18	3.172673739	STA18	5.829469513	0.999621	326.8927402
M19	3.186323744	STA19	5.863290965	0.999597	327.4083178
M20	3.172630517	STA20	5.849904436	0.999608	327.1256065
M21	3.191098494	STA21	5.879646339	0.999625	326.7702927
M22	3.204814588	STA22	5.914483243	0.999589	327.0379383
M23	3.192389206	STA23	5.929740015	0.999536	325.9832844
M24	3.205087637	STA24	5.968127841	0.999506	324.4055384
M25	3.229665337	STA25	6.005163162	0.999511	324.750935
M26	3.260053499	STA26	6.0472667432	0.999523	325.5846946
M27	3.283665334	STA27	6.07752117	0.999555	327.537197
M28	3.319182705	STA28	6.122160683	0.999569	328.262492
M29	3.343575152	STA29	6.162276934	0.999557	327.5105332
M30	3.377146891	STA30	6.203504087	0.999561	327.7860213
M31	3.341076859	STA31	6.178816273	0.999583	329.2402598
M32	3.346509289	STA32	6.180273835	0.999645	332.0426925
M33	3.386265975	STA33	6.233086011	0.99966	332.5282796
M34	3.420702728	STA34	6.280831806	0.999683	333.2787456
M35	3.442085217	STA35	6.285816259	0.999741	334.5544322
M36	3.498008064	STA36	6.345933758	0.999758	334.8398824
M37	3.550338327	STA37	6.399916443	0.999764	334.9558232
M38	3.614322329	STA38	6.463858512	0.999772	335.1406834
M39	3.661430067	STA39	6.522329618	0.999807	335.9587273
M40	3.704199486	STA40	6.576050363	0.999845	336.7564858

	M		STDV	P	Accruac
M41	3.734751288	STA41	6.622449611	0.999908	337.7164538
M42	3.713709904	STA42	6.614729546	1.00079	337.8765556
M43	3.677031966	STA43	6.441319538	1.000288	334.0604948
M44	3.706162991	STA44	6.495438147	1.000386	331.826256
M45	3.707774299	STA45	6.529898532	1.000563	327.1380752
M46	3.70640384	STA46	6.566082361	1.000712	322.6929634
M47	3.581709166	STA47	6.364125294	1.000718	322.6646367
M48	3.502155404	STA48	6.329466436	1.001101	311.2275153
M49	3.47224854	STA49	6.343035213	1.001351	302.9752931
M50	3.1932278	STA50	5.885078541	1.000173	289.1473821
M51	2.941764296	STA51	5.48594924	1.002689	252.9333331
M52	2.80124326	STA52	5.378817034	1.003223	233.9261006
M53	2.658185543	STA53	5.160324364	1.003736	217.185023
M54	2.480248376	STA54	4.827594539	1.004429	196.9391641
M55	2.342274511	STA55	4.692711501	1.005219	175.9037079
M56	2.236877357	STA56	4.539454998	1.004562	189.9974529
M57	2.095821699	STA57	4.236989877	1.00435	194.1080861
M58	1.953636166	STA58	4.031572326	1.005743	170.5162159
M59	1.71467758	STA59	3.518998754	1.006063	165.7221877
M60	1.555942073	STA60	2.971276631	1.004988	177.6992628
M61	1.508768576	STA61	2.923104101	1.00482	179.6725963
M62	1.443544863	STA62	2.858775307	1.006199	168.349762
M63	1.406116899	STA63	2.853909283	1.005558	172.5602708
M64	1.350708513	STA64	2.773337573	1.004772	176.1893157
M65	1.214416234	STA65	2.412597062	1.007731	165.2364818
M66	1.186470742	STA66	2.320034167	1.010303	166.2422972
M67	1.10992398	STA67	2.227893616	1.009481	164.8366538
M68	1.035733157	STA68	2.006945252	1.009481	165.0412983
M69	1.011022504	STA69	1.929523696	1.012653	171.8432658
M70	0.924453611	STA70	1.761582861	1.012168	170.661807
M71	0.920860173	STA71	1.762228438	1.013837	174.9513117
M72	0.827386263	STA72	1.59176215	1.011326	168.3767595
M73	0.831272666	STA73	1.619000237	1.011019	167.5847145
M74	0.715938045	STA74	1.534314484	0.998567	140.3417621
M75	0.726871902	STA75	1.544055275	0.994854	144.176684
M76	0.618994094	STA76	1.180758719	0.998377	140.7395778
M77	0.653417272	STA77	1.193202858	0.993365	147.2940717
M78	0.683779574	STA78	1.210389577	0.989559	154.7185143
M79	0.692192554	STA79	1.215513224	0.988508	159.2759468
M80	0.717735569	STA80	1.251580256	0.988572	162.2082387
				1.013837	
				0.988508	
				0.004649	
				T=(Max-/MIN)/STD	
				5.448298	


0.975
0.98
0.985
0.99
0.995
1
1.005
1.01
1.015
1.02
1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20 21 22 23 24 25 26 27 28 29 30 31 32 33 34 35 36 37 38 39 40 41 42 43 44 45 46 47 48 49 50 51 52 53 54 55 56 57 58 59 60 61 62 63 64 65 66 67 68 69 70 71 72 73 74 75 76 77 78 79 80
p

1- If $E(\varepsilon)\neq 0$, then the estimation of the intercept will be biased and probably the slope estimation will be biased especially in models in which the intercept is absent. In elementary econometrics books, it is proved that there is a negative covariance between the intercept and the slope.

2- ρ may be sensitive to residuals or other variables in a psychological manner. For example, mass perceptions in the stock market can change parameters when the stock market goes to highly expansionary or deeply recessionary states. This statement is more relevant than normal situations. This effect of the variables on ρ is an indirect mean, while the first one is mechanical and direct. Another example of the effects of ε (as a representative of shocks) on ρ are the technology changes that can convey ρ from a stationary position to a nonstationary one.

In the method of determining the magnitude of the above factors, we generated random numbers between zero and one for 10000 observations.

The second time, we used random numbers with zero mean. In this line, we let y_{t-1}equal natural numbers from 1 to 10000 and $y_t = 0.6y_{t-1}$+Ran. Then we ran $y_t \sim y_{t-1}$ and recorded ρ. The second time, 10 Ran were used instead of Ran for ε and the later regressions ran with coefficients according to table 9-A. When errors rose to 1000 times, the estimated ρ rose to 0.674883 whereas the true value is 0.6. In the case that the mean of errors is almost zero (0.02032 in table 9-A), ρ is estimated 0.601634, instead of 0.6 while errors rose by 1000 times of their initial values. This shows that when $E(\varepsilon_t)\neq 0$ the direct effect of ε_t as a proxy of shocks is important, otherwise it is intangible. These biases will be decreased in a model that has a constant 9-B. Therefore the volatility of ρ is not mainly because of estimation bias, or other effects. The indirect effect of ε_t or another variable is responsible for the nonconstancy of ρ .

The Tehran stock exchange total price index has a non-zero mean of errors which can be one of the factors that causes the nonconstancy of ρ , but it is not so important, because the mean of residuals in 80 regression for various samples don't vary significantly. Furthermore the presence of the constant prevents a badly biased estimation of ρ . Then, as the sample size varies, the main determinant of the variation of "ρ" is mass psychology or state of mind of investors. The counterpart of these factors in nonbehaviorial phenomena is technological change.

Coeffcient \ size of Noice	ran	10ran	20ran	30ran	40ran	50ran	100ran	200ran	500ran	1000ran
ρ with positive Noise		0.600749		0.602246		0.603744		0.614911		0.674883
	0.60075		0.601498		0.602995		0.607488		0.637442	
	Mean = 0.502747									
ρ with almost Zero Mean Noise	0.600002		0.600033		0.600065		0.600163		0.600817	
		0.600016		0.600049		0.600082		0.600327		0.601634
	Mean = .002032									

A

ran	ran	10ran	20ran	30ran	40ran	50ran	100ran	200ran	500ran	1000ran
C	0.5113385		10.26769		20.53539		51.33847		256.6923	
ρ	0.599998		0.599957		0.599915		0.599787		0.598936	
C		5.133844		15.40154		25.66923		102.6769		513.3847
ρ		0.599979		0.599936		0.599894		0.599574		0.597872
C	-0.024713		-0.494256		-0.988512		-2.471279		-12.35640	
ρ	0.600005		0.600107		0.600214		0.600534		0.602671	
zero Mean C		-2.247128		-0.741384		-1.235640		-4.942558		-24.71279
zero Mean ρ		0.600053		0.600160		0.600267		0.601074		0.595341

B

It is time to pay attention to the catastrophe and chaos theories for finding stable and unstable equilibria.

The regression equations of the first derivatives of the catastrophe equations in page 10 are as follows:

Fold $D(p) = \alpha + \beta_1 \rho_2 + \beta_2 M + \zeta_t$ *(when the mean of the absolute residuals is a control variable)*

Fold $D(p) = \alpha + \beta_1 \rho_2 + \beta_2 STDV + \zeta_t$ *(when the standard derivation of the residuals is a control variable)*

cusp $D(p) = \alpha + \beta_1 \rho_3 + \beta_2 M + \beta_3 \rho.STDV + \zeta_t$

cusp $D(p) = \alpha + \beta_1 \rho_3 + \beta_2 STDV + \beta_3 M + \zeta_t$

The results of the estimation are presented in 10-A to 10-D, respectively. In these outputs, "ρ" is the coefficient of various regressions of different samples, s ρ is the square of ρ, cρ is the cubic of ρ, EE is ρ.M, FF is ρ.STDV. As can be seen from the outputs behavior of ρ, it is Fold type, and the Cusp equation is not so relevant to this process.

An educated guess about the relation between the residuals and ρ_i in each equation was the effect of the Accrual of the Residuals (ACR) instead of the mean (M) and standard error (STDV) of it. The E views output(10-E & 10-F) shows that in this case, Fold is also a relevant functional form.

These results imply that the change of ρ in various samples is not smooth and has some discontinuities. The sign of control variables is not negative, that is because of the negative sign of intercept that in theory is not considered in catastrophe equations.

Another theory that can help us to explain that the behavior of ρ in different samples is not normal is the chaos theory. One of the commonly used equations in chaos is the logistic map. Equations that we used for the test of ρ's behavior are:

$$\rho_t = \alpha + 4\beta_1\rho_{t-1}(1-\rho_{t-1}) + \beta_2\rho_{t-1} + \zeta_t \quad (1)$$

$$\rho_t = \alpha + 4\beta_1\rho_{t-1}(1-\rho_{t-1}) + \zeta_t \quad (2)$$

$$\rho_t = \alpha + \beta_1\rho_{t-2} + \beta_2\rho^2_{t-1} + \zeta_t \quad (3)$$

Results in 10-G to 10-I show that the change of behavior is substantial and chaos behavior is relevant. High and stable R-2 in each of the equations and very good D-W in 15-c shows the robustness of the results.

A

Dependent Variable: D(P)
Method: Least Squares
Date: 05/12/01 Time: 14:19
Sample(adjusted): 2 80
Included observations: 79 after adjusting endpoints

Variable	Coefficient	Std. Error	t-Statistic	Prob.
C	-0.087698	0.021662	-4.048396	0.0001
SP	0.085650	0.021450	3.992949	0.0001
M	0.000621	0.000189	3.291761	0.0015

R-squared	0.216653	Mean dependent var	-000152
Adjusted R-squared	0.196038	S.D. dependent var	0.001899
S.E. of regression	0.001703	Akaike info criterion	-9.875547
Sum squared resid	0.000220	Schwarz criterion	-9.785568
Log likelihood	393.0841	F-statistic	10.50976
Durbin-Watson stat	1.910070	Prob(F-statistic)	0.000093

B

Dependent Variable: D(P)
Method: Least Squares
Date: 05/12/01 Time: 14:20
Sample(adjusted): 2 80
Included observations: 79 after adjusting endpoints

Variable	Coefficient	Std. Error	t-Statistic	Prob.
C	-0.085755	0.021522	-3.984573	0.0002
SP	0.083645	0.021315	3.924153	0.0002
STDV	0.000353	0.000108	3.266717	0.0016

R-squared	0.215168	Mean dependent var	-000152
Adjusted R-squared	0.194515	S.D. dependent var	0.001899
S.E. of regression	0.001705	Akaike info criterion	-9.873654
Sum squared resid	0.000221	Schwarz criterion	-9.783675
Log likelihood	393.0093	F-statistic	10.41801
Durbin-Watson stat	1.921136	Prob(F-statistic)	0.000100

C

Dependent Variable: D(P)
Method: Least Squares
Date: 05/12/01 Time: 14:18
Sample(adjusted): 2 80
Included observations: 79 after adjusting endpoints

Variable	Coefficient	Std. Error	t-Statistic	Prob.
C	-0.059352	0.015238	-3.895073	0.0002
CP	0.057304	0.015111	3.792328	0.0003
M	-0.000698	0.002041	0.341875	0.7334
FF	-4.26E-05	0.001168	-0.036465	0.9710

R-squared	0.217218	Mean dependent var	-000152
Adjusted R-squared	0.185906	S.D. dependent var	0.001899
S.E. of regression	0.001714	Akaike info criterion	-9.850952
Sum squared resid	0.000220	Schwarz criterion	-9.730980
Log likelihood	393.1126	F-statistic	6.937355
Durbin-Watson stat	1.908123	Prob(F-statistic)	0.000349

·D

Dependent Variable: D(P)
Method: Least Squares
Date: 05/12/01 Time: 14:23
Sample(adjusted): 2 80
Included observations: 79 after adjusting endpoints

Variable	Coefficient	Std. Error	t-Statistic	Prob.
C	-0.059376	0.014883	-3.989471	0.0002
CP	0.057348	0.014729	3.893627	0.0002
STDV	-0.000146	0.001186	-0.122805	0.9026
EE	0.000878	0.002074	0.423400	0.6732

R-squared	0.217598	Mean dependent var	-000152
Adjusted R-squared	0.186301	S.D. dependent var	0.001899
S.E. of regression	0.001713	Akaike info criterion	-9.851438
Sum squared resid	0.000220	Schwarz criterion	-9.731466
Log likelihood	393.1318	F-statistic	6.952864
Durbin-Watson stat	1.904702	Prob(F-statistic)	0.000343

E

Dependent Variable: D(P)
Method: Least Squares
Date: 05/12/01 Time: 10:17
Sample(adjusted): 2 80
Included observations: 79 after adjusting endpoints

Variable	Coefficient	Std. Error	t-Statistic	Prob.
C	-0.090640	0.022493	-4.029692	0.0001
SP	0.088026	0.022160	3.972274	0.0002
ACR	8.46E-06	2.86E-06	2.954651	0.0042

R-squared	0.197185	Mean dependent var	-0.000152
Adjusted R-squared	0.176058	S.D. dependent var	0.001899
S.E. of regression	0.001724	Akaike info criterion	-9.850999
Sum squared resid	0.000226	Schwarz criterion	-9.761019
Log likelihood	392.1144	F-statistic	9.333422
Durbin-Watson stat	1.850693	Prob(F-statistic)	0.000237

F

Dependent Variable: D(P)
Method: Least Squares
Date: 05/12/01 Time: 10:16
Sample(adjusted): 2 80
Included observations: 79 after adjusting endpoints

Variable	Coefficient	Std. Error	t-Statistic	Prob.
C	-0.059241	0.014994	-3.951086	0.0002
CP	0.057173	0.014635	3.906501	0.0002
M	0.000618	0.000450	1.371908	0.1742
HH	9.58E-08	6.76E-06	0.014176	0.9887

R-squared	0.217206	Mean dependent var	0.000152
Adjusted R-squared	0.185894	S.D. dependent var	0.001899
S.E. of regression	0.001714	Akaike info criterion	-9.850937
Sum squared resid	0.000220	Schwarz criterion	-9.730965
Log likelihood	393.1120	F-statistic	6.936874
Durbin-Watson stat	1.909392	Prob(F-statistic)	0.000349

G

Dependent Variable: P
Method: Least Squares
Date: 05/13/01 Time: 10:26
Sample(adjusted): 2 80
Included observations: 79 after adjusting endpoints
P= C(1)+4*C(2)*P(-1)*(1-P(-1))+C(3)*P(-1)

	Coefficient	Std. Error	t-Statistic	Prob.
C(1)	-10.56834	5.842457	-1.808886	0.0744
C(2)	2.637200	1.452822	1.815226	0.0734
C(3)	11.56839	5.842534	1.980030	0.0513

R-squared	0.843129	Mean dependent var	1.001507
Adjusted R-squared	0.839001	S.D. dependent var	0.004677
S.E. of regression	0.001877	Akaike info criterion	-9.681231
Sum squared resid	0.000268	Schwarz criterion	-9.591252
Log likelihood	385.4086	Durbin-Watson stat	1.815895

H

Dependent Variable: P
Method: Least Squares
Date: 05/13/01 Time: 10:28
Sample(adjusted): 2 80
Included observations: 79 after adjusting endpoints
P= C(1)+4*C(2)*P(-1)*(1-P(-1))

	Coefficient	Std. Error	t-Statistic	Prob.
C(1)	0.999898	0.000230	4346.406	0.0000
C(2)	-0.239333	0.012123	-19.74259	0.0000

R-squared	0.835037	Mean dependent var	1.001507
Adjusted R-squared	0.832894	S.D. dependent var	0.004677
S.E. of regression	0.001912	Akaike info criterion	-9.656248
Sum squared resid	0.000282	Schwarz criterion	-9.596262
Log likelihood	383.4218	Durbin-Watson stat	1.709469

I

Dependent Variable: P
Method: Least Squares
Date: 05/13/01 Time: 11:18
Sample(adjusted): 3 80
Included observations: 78 after adjusting endpoints
P= C(1)+C(2)*P(-2)+C(3)*(P(-1))^2

	Coefficient	Std. Error	t-Statistic	Prob.
C(1)	0.612562	0.074128	8.263608	0.0000
C(2)	-0.161357	0.121647	-1.326439	0.1887
C(3)	0.548761	0.057126	9.606221	0.0000

R-squared	0.839388	Mean dependent var	1.001519
Adjusted R-squared	0.835105	S.D. dependent var	0.004707
S.E. of regression	0.001911	Akaike info criterion	-9.644464
Sum squared resid	0.000274	Schwarz criterion	-9.553822
Log likelihood	379.1341	Durbin-Watson stat	1.992078

4- The Estimation of the VRDE models

When ρ is not a constant estimation, it will be different from the constant one. We focus on the estimation procedure as follows:

$$y_t = \rho_1(\varepsilon_t, \varepsilon_{t-1}, \varepsilon_{t-2}, \ldots, \varepsilon_{t-k}) y_{t-1} + \varepsilon_t$$

or in general

$$y_t = \sum_{i=1}^{P} \rho_i(\varepsilon_{t-k}, \ldots, \varepsilon_{t-i+1}) y_{t-i} + \varepsilon_t$$

For simplicity, we get the mean and variance of the AR(1) process and will see generalization in the appendix. In compact form,

$y_t = \rho_1(\vec{L}\varepsilon_t) y_{t-1} + \varepsilon_t, L$ Is lag operator

$$\begin{aligned}
y_{t-1} &= \rho_1(\vec{L}\varepsilon_t)\left[\rho_1(\vec{L}\varepsilon_t) y_{t-2} + \varepsilon_{t-1}\right] \\
&= \rho_1^2(\vec{L}\varepsilon_t) y_{t-2} + \rho_1(\vec{L}\varepsilon_t)\varepsilon_{t-1} + \varepsilon_t \\
&= \rho_1^2(\vec{L}\varepsilon_t)\left[\rho_1^2(\vec{L}\varepsilon_t) y_{t-3} + \varepsilon_{t-2}\right] + \rho_1(\vec{L}\varepsilon_t) + \varepsilon_t \\
&= \rho_1^3(\vec{L}\varepsilon_t) y_{t-3} + \rho_1^2(\vec{L}\varepsilon_t)\varepsilon_{t-2} + \rho_1(\vec{L}\varepsilon_t)\varepsilon_{t-1} + \varepsilon_t \\
&\vdots \\
&= \rho_1^{k+1}(\vec{L}\varepsilon_t) y_{t-k-1} + \rho_1^{k}(\vec{L}\varepsilon_t)\varepsilon_{t-k} + \ldots + \rho_1(\vec{L}\varepsilon_t)\varepsilon_{t-1} + \varepsilon_t \\
&= \rho_1^{k+1}(\vec{L}\varepsilon_t) y_{t-k-1} + \sum_{i=0}^{k} \rho_1^{i}(\vec{L}\varepsilon_t)\varepsilon_{t-i}
\end{aligned}$$

Then $E(y_t) = E\left[\rho_1(\vec{L}\varepsilon_t) y_{t-k-1}\right] + E\sum_{i=0}^{k} \rho_1^{i}(\vec{L}\varepsilon_t)\varepsilon_{t-i}$

Without serial correlation, none of the elements of $\vec{L}\varepsilon_t$ can be correlated with Y_{t-k-1} and with the approximation of $E[F(x)] \approx F[E(x)]$, the first term in the right hand of equation will be $AE(Y_{t-k-1})$. Where A is $P_{1k+1}(0)$

with the assumption of $\varepsilon_t \sim$ i.i.d$(0,\sigma^{\varepsilon}_{2})$.For the second term in the right hand of the equation, some parts will be zero and the others result in $\sum_{i=0}^{k} \varepsilon^{i}_{t-i} = B$

Therefore :

$$E(y_t) = AE(y_{t-k-1}) + B$$

$$\mathrm{var}(y_t) = \mathrm{var}\, \rho_1^{k+1}(\vec{L}\varepsilon_t) y_{t-k-1} + \mathrm{var} \sum_{i=0}^{k} \rho_1^{i}(\vec{L}\varepsilon_t)\varepsilon_{t-i}$$

$$+ 2\,\mathrm{cov}\left[\rho_1^{k+1}(\vec{L}\varepsilon_t) y_{t-k-1}, \sum_{i=0}^{k} \rho^{i}(\vec{L}\varepsilon_t)\varepsilon_{t-i}\right]$$

This shows that the estimation of the VRDE models with OLS or ML are not only difficult but also strictly opposed to our fundamental opposed assumption of the variable roots.

The estimation method that can be used in this case is 2SLS. The applications of this method are illustrated partly in previous sections. However in doing so, one should change the sample range and estimate different ρ_s and then run ρ_s on mean of the residuals of each sample with various functional forms. The resulting equation will be a variable root of the primary equation. If the researcher recognizes that the residuals are not good proxies for shocks, he can use another variable. Also it is possible that ρ_s not be functions of the shocks that in this case will be used as explanatory relevant variables. If none of the cases above occur, ρ_s can be regressed on their lags:

$$\rho_t = \alpha + \sum_{i=1}^{n} \beta_i \alpha_{t-i} + v_t, v_t \sim \mathrm{i.id}(0,\sigma^2)$$

This equation will be a variable root for the primal equation, i.e $y_t = \rho_1$ $y_{t-1} + \varepsilon_t$. The results of the estimation for ρ are presented in 12-A . Again, ρ is the coefficient of index on the index(-1) for a different number of observations.

5- Conclusion and further studies

Economic series may be modeled by variable root difference equations; such that for some observations, a root is proper which is improper for other observations (or periods with the same number of observations). In other words, ρ is AR(1) process. For example, it is not constant in respect to number of observations or different time periods. Similar to this debate is the STUR process in which ρ_s ocillates around one (unit) stochastically. The difference of our discussion is that ρ may be 0.5 for one period and 0.8 for another one and not necessarily around one. Although financial series are more similar to STUR then VRDE, the possibility of the existence of this process is not challengeless.

In addition to the reasoning mentioned above the change of a root from stationary to nonstationary is not a smooth change and there are probably discontiniuties in this change. Our results show a weak form of Fold type catastrophe and chaotic behavior for ρ_s in samples with variable sizes.

Our brief suggestions are studying about variable root difference equations and the determination of conditions affecting them.

Appendix

We want to show AR(2) and AR(P) forms when ρ is a function of ε_t and it's lags. First AR(2):

$$y_t = \rho_1(\vec{L}\varepsilon_t)y_{t-1} + \rho_2(\vec{L}\varepsilon_t)y_{t-2} + \varepsilon_t$$

$$\rho_1^2(\vec{L}\varepsilon_t)y_{t-2} + \rho_1\,\rho_2(\vec{L}\varepsilon_t)y_{t-3} + \rho_2^2(\vec{L}\varepsilon_t)y_{t-4} + \rho_1\,(\vec{L}\varepsilon_t)| + \rho_2(\vec{L}\varepsilon_t)\varepsilon_{t-2} + \varepsilon_t$$

after some manipulations

$$= \left[\rho_1(\vec{L}\varepsilon_t) + \rho_2(\vec{L}\varepsilon_t)\right]^n y_{t-(\text{power.index for }\rho_1 + \text{power.index for }\rho_2)} + \varepsilon$$

$$+ \left[\rho_1(\vec{L}\varepsilon_t) + \rho_2(\vec{L}\varepsilon_t)\right]^{n-1} \varepsilon_{t-(\text{power.index for }\rho_1 + \text{power.index for }\rho_2)}$$

or simply

$$y_t = \sum_{i=0}^{n} \begin{pmatrix} n \\ k \end{pmatrix} \rho_1^{n-k}(\vec{L}\varepsilon_t) + \rho_2^k(\vec{L}\varepsilon_t) y_{t-(n+k)}$$

$$= \sum_{i=1}^{n-1}\sum_{k=0}^{n-1} \begin{pmatrix} n-i \\ k \end{pmatrix} \rho_1^{n-k-i}\,\rho_2^k(\vec{L}\varepsilon_t)\;\rho_2^k(\vec{L}\varepsilon_t)\;\varepsilon_{t-(n+k-i)}$$

now for AR(P)

$$y_t = \sum_{i=1}^{p} \rho_i(\vec{L}\varepsilon_t) + \varepsilon_t$$

$$= \left[\sum_{i=1}^{p} \rho_i(\vec{L}\varepsilon_t)\right]^n y_{t-(\sum_{i=1}^{p} \text{index.power for}\rho_i)}$$

$$+\left[\left[\sum_{i=1}^{p} \rho_i (\vec{L}\varepsilon_t)\right]^{n-1} \varepsilon_t)\right]^n y_{t-(\sum_{i=1}^{p} \text{index.power for} \rho_i)} + \ldots$$

$$+\left[\sum_{i=1}^{p} \rho_i (\vec{L}\varepsilon_t)\right]\varepsilon_{t-(\sum_{i=1}^{p} \text{index.power for} \rho_i)} + \varepsilon_t$$

Or similary:

$$y_t = \sum_{k=0}^{n} \frac{n!}{\prod_{j=1}^{k}} \prod_{j=1} \rho_j^{nj} (\vec{L}\varepsilon_t) + y_{t-(n+nj)}$$

$$\sum_{k=0}^{n-1} \sum_{i=1}^{n-1} \frac{(n-i)!}{\prod_{j=1}^{k}} \prod_{i=1}^{k} \rho_j^{nj-i} (\vec{L}\varepsilon_t).\varepsilon_{t-(n+nj-i)}$$

Obs	P	RESIS	SP	STDV
1	1.000568	NA	1.001136	5.892624
2	1.000580	0.000138	1.001160	50889794
3	1.000534	8.08E-05	1.001068	5.873006
4	1.000610	0.000201	1.001220	5.879186
5	1.000623	0.000141	1.001246	5.885621
6	1.000417	-7.74E-05	1.000834	5.866352
7	1.000322	2.50E-05	1.000644	5.866056
8	1.002310	0.002104	1.004625	5.844325
9	1.000233	-0.001881	1.000466	5.775928
10	1.000272	0.000151	1.000544	5.774043
11	1.000170	1.19E-05	1.000340	5.771934
12	1.000137	7.66E-05	1.000274	5.789319
13	0.999977	-5.18E-05	0.999954	5.748186
14	0.999860	-1.56E-05	0.999720	5.757737
15	0.999810	4.64E-05	0.999620	5.783936
16	0.999763	4.72E-05	0.999526	5.808220
17	0.999669	-1.82E-06	0.999338	5.804612
18	0.999621	4.01E-05	0.999242	5.829470
19	0.999597	6.20E-05	0.999194	5.863291
20	0.999608	9.60E-05	0.999216	5.849904
21	0.999625	0.000102	0.999250	5.879640
22	0.999589	5.02E-05	0.999178	5.914483
23	0.999536	3.17E-05	0.999072	5.929740
24	0.999506	5.23E-05	0.999012	5.963128
25	0.999511	8.60E-05	0.999022	6.005163
26	0.999523	9.33E-05	0.999046	6.047267
27	0.999555	0.000114	0.999110	6.077521
28	0.999569	9.72E-05	0.999138	6.122161
29	0.999557	7.18E-05	0.999114	6.162277
30	0.999561	8.73E-05	0.999122	6.203504
31	0.999583	0.000105	0.999166	6.178816
32	0.999645	0.000146	0.999290	6.180274
33	0.999660	0.000102	0.999320	6.233086
34	0.999683	0.000111	0.999366	6.280832
35	0.999741	0.000147	0.999482	6.285816
36	0.999758	0.000108	0.999516	6.345934
37	0.999764	9.80E-05	0.999528	6.399916
38	0.999772	0.000100	0.999544	6.463859
39	0.999807	0.000128	0.999614	6.522330
40	0.999845	0.000132	0.999690	6.576050
41	0.999908	0.000159	0.999816	6.622450
42	1.000790	0.000980	1.001581	6.614730
43	1.000288	-0.000367	1.000576	6.441320
44	1.000563	0.000213	1.000772	6.495438
45	1.000563	0.000296	1.001126	6.529899

Obs	P	RESIS	SP	STDV
46	1.000712	0.000275	1.001425	6.566082
47	1.000718	0.000138	1.001437	6.364125
48	1.001101	0.000515	1.002203	6.329466
49	1.001351	0.000398	1.002704	6.343035
50	1.000173	-0.001020	1.000346	5.885079
51	1.002689	0.002626	1.005385	5.485949
52	1.003223	0.000744	1.006455	5.378817
53	1.003736	0.000743	1.007486	5.160324
54	1.004429	0.000941	1.008878	4.827595
55	1.005219	0.001063	1.010465	4.692712
56	1.004562	-0.000358	1.009145	4.539455
57	1.004350	6051E-05	1.008719	4.236990
58	1.005743	0.001663	1.011519	4.031572
59	1.006063	0.000636	1.012163	3.518999
60	1.004988	-0.000749	1.010001	2.971277
61	1.004820	0.000123	1.009663	2.923104
62	1.006199	0.001665	1.012436	2.858775
63	1.005558	-0.000311	1.011147	2.853909
64	1.004772	-0.000476	1.009567	2.773338
65	1.007731	0.003243	1.015522	2.412597
66	1.010303	0.002947	1.020712	2.320034
67	1.009481	-0.000382	1.019052	2.227894
68	1.009481	0.000421	1.019052	2.006945
69	1.012653	0.003593	1.025466	1.929524
70	1.012168	3.95E-06	1.024484	1.761583
71	1.013837	0.002149	1.027865	1.762228
72	1.011326	-0.002002	1.022780	1.591762
73	1.011019	0.000156	1.022159	1.19000
74	0.998567	-0.11996	0.997136	1.534314
75	0.994854	-0.003674	0.989734	1.544055
76	0.998377	0.003380	0.996757	1.180759
77	0.993365	-0.004981	0.986774	1.193203
78	0.989559	-0.004029	0.979227	1.210390
79	0.988508	-0.001499	0.977148	1.215513
70	0.988572	-0.000450	0.977275	1.251580
81	NA	NA	NA	NA
82	NA	NA	NA	NA
83	NA	NA	NA	NA
84	NA	NA	NA	NA
85	NA	NA	NA	NA
86	NA	NA	NA	NA
87	NA	NA	NA	NA
88	NA	NA	NA	NA
89	NA	NA	NA	NA
90	NA	NA	NA	NA

Obs	ACR	CP	EE	FF	HH	M
1	238.6914	1.001705	3.468134	5.895671	238.8270	3.466166
2	237.1283	1.001741	3.458023	5.893210	237.2659	3.456018
3	242.8879	1.001603	3.432547	5.876142	243.0176	3.430715
4	233.6635	1.001831	3.420438	5.882772	233.8060	3.418352
5	232.1421	1.001870	3.415002	5.889287	232.2867	3.412876
6	256.4185	1.001252	3.381182	5.868798	256.5255	3.376773
7	267.7732	1.000966	3.361006	5.867945	267.8594	3.359924
8	278.1530	1.006946	3.334048	5.857825	278.7955	3.326364
9	277.9329	1.000699	3.274877	5.777274	277.9976	3.244114
10	273.6072	1.000816	3.256377	5.775613	273.6816	3.255491
11	284.4436	1.000510	3.241094	5.772915	284.4919	3.240543
12	287.7730	1.000411	3.242675	5.790112	287.8124	3.242230
13	303.0779	0.999931	3.201192	5.748054	303.0709	3.201266
14	312.9997	0.999580	3.186242	5.756930	312.9559	3.186688
15	316.7813	0.999430	3.192801	5.782837	316.7211	3.193407
16	319.9758	0.999289	3.195658	5.806843	319.9000	3.196416
17	325.0947	0.999007	3.167031	5.802691	324.9871	3.168080
18	326.8927	0.998863	3.171471	5.827260	326.7688	3.172674
19	327.4083	0.998791	3.185040	5.860928	327.2764	3.186324
20	327.1256	0.998824	3.171387	5.847611	326.9974	3.172631
21	326.7703	0.998875	3.189902	5.887441	326.6478	3.191098
22	327.0379	0.998768	3.203497	5.912052	326.9035	3.204815
23	325.9833	0.998609	3.190908	5.956989	325.8320	3.192389
24	324.4055	0.998519	3.203504	5.965180	324.2453	3.205088
25	324.7509	0.998534	3.228086	6.002227	324.5921	3.229665
26	325.5847	0.998570	3.258498	6.044383	325.4294	3.260053
27	327.5372	0.998666	3.282204	6.074817	327.3914	3.283665
28	328.2625	0.998708	3.317752	6.119522	328.1210	3.319183
29	327.5105	0.998672	3.342094	6.159547	327.3654	3.343575
30	327.7860	0.998684	3.375664	6.200781	327.6421	3.377147
31	329.2403	0.998750	3.339684	6.176240	329.1030	3.341077
32	332.0427	0.998935	3.345321	6.178080	331.9248	3.346509
33	332.5283	0.998980	3.385115	6.230967	332.4152	3.386266
34	333.2787	0.999049	3.419618	6.278841	333.1731	3.420703
35	334.5544	0.999223	3.441194	6.284188	334.4678	3.442085
36	334.8399	0.999274	3.497162	6.344398	334.7589	3.498008
37	334.9558	0.999292	3.549500	6.398406	334.8768	3.550338
38	335.1407	0.999316	3.613498	6.462385	335.0643	3.614322
39	335.9587	0.999421	3.660723	6.521071	335.8939	3.661430
40	336.7565	0.999535	3.703625	6.575031	336.7043	3.704199
41	337.7165	0.999724	3.734408	6.621840	337.6854	3.734751
42	337.8766	1.002372	3.716644	6.619955	338.1435	3.713710
43	334.0605	1.000864	3.678091	6.443175	334.1567	3.677032
44	331.8263	1.001158	3.707594	6.497945	331.9543	3.706163
45	327.1381	1.001690	3.709862	6.533575	327.3223	3.707774

Obs	ACR	CP	EE	FF	HH	M
46	322.6930	1.002138	3.709043	6.570757	322.9227	3.706404
47	322.6649	1.002156	3.584281	6.368695	322.8966	3.581709
48	311.2275	1.003307	3.506011	6.336435	311.5702	3.502155
49	302.9753	1.004058	3.476940	6.351605	303.3846	3.472249
50	289.1474	1.000518	3.193780	5.886095	289.1973	3.193228
51	252.9333	1.008089	2.949675	5.500701	253.6135	2.941764
52	233.9261	1.009700	2.810272	5.396153	234.6800	2.801243
53	217.1850	1.011250	2.668117	5.179603	217.9964	2.658186
54	196.9392	1.013346	2.491233	4.848976	197.8114	2.480248
55	175.9037	1.015739	2.354499	4.717203	176.8217	2.342275
56	189.9975	1.013749	2.247082	4.560164	190.8642	2.236877
57	194.1081	1.013107	2.104939	4.255421	194.9525	2.095822
58	170.5162	1.017328	1.964856	4.054726	171.4955	1.953636
59	165.7222	1.018300	1.725074	3.540334	166.7270	1.714678
60	177.6993	1.015039	1.563703	2.986097	178.5856	1.555942
61	179.6726	1.014530	1.516041	2.937193	180.5386	1.508769
62	168.3498	1.018713	1.452493	2.876497	169.3934	1.443545
63	172.5603	1.016767	1.416949	2.869771	173.5194	1.409117
64	176.1893	1.014384	1.357154	2.786572	177.0301	1.350709
65	165.2365	1.023373	1.223805	2.431249	166.5139	1.214416
66	166.2423	1.031229	1.198695	2.343937	167.9551	1.186471
67	164.8367	1.028714	1.120447	2.249016	166.3995	1.109924
68	165.0413	1.028714	1.045553	2.025973	166.6061	1.035733
69	171.8433	1.038441	1.023815	1.953938	174.0176	1.011023
70	170.6618	1.036950	0.935702	1.783018	172.7384	0.924454
71	174.9513	1.042088	0.933602	1.786612	177.3721	0.920860
72	168.3768	1.034364	0.836757	1.609790	170.2838	0.872386
73	167.5847	1.033423	0.840432	1.636840	169.4313	0.831273
74	140.3418	0.995707	0.714912	1.532116	140.1407	0.715938
75	144.1767	0.984641	0.723131	1.536110	143.4348	0.726872
76	140.7396	0.995139	0.617989	1.178842	140.5112	0.618994
77	147.2941	0.980227	0.649082	1.185286	146.3168	0.653417
78	154.7185	0.969003	0.676640	1.197752	153.1031	0.683780
79	159.2759	0.965919	0.684238	1.201545	157.4455	0.692193
80	162.2082	0.966106	0.709533	1.237277	160.3545	0.717736
81	NA	NA	NA	NA	NA	NA
82	NA	NA	NA	NA	NA	NA
83	NA	NA	NA	NA	NA	NA
84	NA	NA	NA	NA	NA	NA
85	NA	NA	NA	NA	NA	NA
86	NA	NA	NA	NA	NA	NA
87	NA	NA	NA	NA	NA	NA
88	NA	NA	NA	NA	NA	NA
89	NA	NA	NA	NA	NA	NA
90	NA	NA	NA	NA	NA	NA

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