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A SUPER ZERO-REST-MASS-EQUATION

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Abstract

Various authors have considered the Zero-rest-mass equation and the contour integral representation of its solutions. Ferber generalized these equations to supertwistor spaces with 2N odd components so that with N=0 we get the standard ungraded twistors of Penrose. In this paper we use the Batchelor theorem to construct the natural super Twistor space with coarse topology with underlying standard twistors. We also introduce a Super zero rest-mass equation (S. Z. R. M) which satifies the standard non- graded Z. R. M. equation when an augmentation map ε is applied. It has been shown that the contour integral defined by Rogers can be used to represent a solution for these equations and these solutions reduce to standard solutions when one applies the ε-map.

Introduction

1) An algebra B is called a $\mathbb{Z}_2$-graded algebra if B can be written as a direct sum of linear subspaces $B=B_0 \oplus B_1$ such that $B_iB_j \subset B_{i+j}$ (mod 2).

An element b in B_i is said to be homogeneous of degree i. Let $|a|$ denote the degree of a homogeneous element a. An element $b \in B_0$ is called even and if $b \in B_1$ then b is called odd. The algebra B is said to be graded commutative if $ab=(-1)^{|a||b|}ba$.

For any of the graded commutative algebra B defined above there is a unique algebra homomorphism $\varepsilon: B \longrightarrow \mathbb{C}$ given by sending the odd generators to zero, $\mathbb{C}$ is the set complex number, ε is called augementation map.

In this paper we consider the Grassmann algebras over complex numbers $\mathbb{C}$, with a finite generators $1, \theta_1, \theta_2, \ldots, \theta_L$ for graded algebra B.

Let M_L denote the set of finite sequences of positive integers $\mu=(\mu_1, \ldots, \mu_k)$ with $1 \leqslant \mu_1 < \ldots < \mu_k \leqslant L$. M_L includes the sequence with no elements, denoted φ. then if for each μ in M_L, $\theta_\mu := \theta_{\mu_1} \cdots \theta_{\mu_k}$ & $\theta_\varphi := 1$, a typical element $b \in B$ may be expressed as

$$b=\sum_{\mu \in M_L} b^\mu \theta_\mu$$

Key words: Zero-rest-mass-equation, non graded Z. R. M.equation

where the coefficient b^μ are complex numbers.

With the norm on B defined by $\|b\|=\sum_{\mu \in M_L} b^\mu$, B is Banach algebra. And B has a topology given by the Bananch space structure on B, [5].

However there is another topology, the coarse topology:

Say a set $U \subset B$ is open in coarse topology on B if $U=\varepsilon^{-1}$ (V) for some open set V in $\mathbb{C}$, this topology is not Hausdorff, [2].

For each pair of positive integers r and s let $E^{r,s}$ denote the cartesian product of r copies of the even part of B and s copies of the odd part;

i.e, $E^{r,s}=(B_0)^r \times (B_1)^s$. $E^{r,s}$ is called supereuclidean space.

If B has a topology, B_0 and B_1 have a topologies as subspaces of B, and $E^{r,s}$ has the topology of a product of topological spaces.

Using the coarse topology for B, this means that a set $U \subset E^{r,s}$ is open in coarse topology if and only if $U=\varepsilon^{-1}$ (V) where V is an open set in $\mathbb{C}^r$ and $\varepsilon: E^{r,s} \longrightarrow \mathbb{C}^r$ is given by $\varepsilon(x_1, \ldots, x_r, \theta_1, \ldots, \theta_s)=(\varepsilon(x_1), \ldots, \varepsilon(x_r))$, [2].

2) Let S denote a choice of a definition of super euclidean space together with a topology and a smooth structure. Define an S supermanifold to be a topological space X together with an atlas $\{U_\alpha, \varphi_\alpha\}$ of S-smooth

hommomorphism from U_α to an open set in some supereuclidean space $E^{r,s}$ such that

$\varphi_\alpha o \varphi_\beta^{-1}$: $\varphi_\beta (U_\alpha \cap U_\beta) \longrightarrow \varphi_\alpha (U_\alpha \cap U_\beta)$ is S-smooth, [2].

3) A complex two-dimensional vector space S with elements α^A on which SL $(2,\mathbb{C})$ acts, is spin- space and the elements are spinors. The complex conjugate vector space $\overline{S}=S'$ has elements $\beta^{A'}$ and we also have the two dual space S^*, S'^* with elements γ_A, $\delta_{A'}$.

We may develop the spinor algebra by analogy with the tensor algebra.

Higher valence spirors are elements of tensor products:

$\varphi^{A...BA'...C'}C...DE'...F' \in S \otimes ... \otimes S \otimes S' \otimes ... \otimes S' \otimes ... \otimes S^* \otimes ... \otimes S^* \otimes S'^* \otimes ... \otimes S'^*$.

We consider basic space of spinors $S = (\mathbb{C}^2, \Gamma$, where Γ is a skew- symmetric nondegenerate complex bilinear form.

Let M_0 be Minkowski space, i.e., $\{\mathbb{R}^4$ equipped with a flat Lorentz metric of signature $(+,-,-,-)\}$ and $x = (x^0, x^1, x^2, x^3)$ be coordinates for M_0. Also let $\{x^{AA'}\}$ be the spinor coordinates of M_0, where $x^{00'} = \frac{1}{2^{1/2}}(x^0 + x^1)$, $x^{10'} = \frac{1}{2^{1/2}}(x^2 - ix^3)$, $x^{01'} = \frac{1}{2^{1/2}}(x^2 + ix^3)$ and $x^{11'} = \frac{1}{2^{1/2}}(x^0 - x^1)$.

Define $\nabla_{AA'} = \frac{\partial}{\partial x^{AA'}}$, $\nabla^{AA'} = \frac{\partial}{\partial x_{AA'}}$

where $x_{AA'}$ are dual variables expressed by

$$x_{AA'} = x^{BB'} \Gamma_{BA} \Gamma_{B'A'}.$$

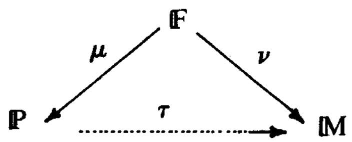
The differential equation on real or complexified Minkowski space

$$\nabla^{AA'} \varphi_{ABC...D} = 0, \ \nabla^{AA'} \psi_{A'B'...D'} = 0$$

are the zero-rest-mass field equations, where $\varphi_{ABC...D'}$ $\psi_{A'B'...D'}$ are symmetric spinors with 2S indices, $s = 1/2, 1, 3/2, ...$ [7].

4) A Twistor space Π is, by definition, a complex vector- space of dimension 4 with an Hermitian form φ of signature $++--$.

Define the flag manifold $\mathbb{F}_{d_1,...,d_r}$ as follows:
$\mathbb{F}_{d_1,...,d_r} := \{(L_1, ..., L_r): L_1 \subset ... \subset L_r$ is a sequence of Linear subspaces of Π with $\dim_{\mathbb{C}} L_j = d_j\}$.
The sets $\mathbb{P} := \mathbb{F}_1 \cong \mathbb{P}_3(\mathbb{C})$, $M := \mathbb{F}_2 \cong G_{2,4}(\mathbb{C})$ and $\mathbb{F} := \mathbb{F}_{12}$ are called the projective twistor space, the compactified complexified Minkowski space, and the correspondence space between $\mathbb{P}$ and $\mathbb{M}$, respectively. These compact complex manifolds are naturally linked by the double fibration.

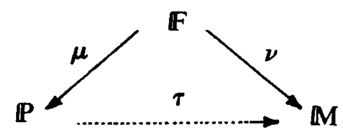


where $\mu(L_1, L_2) = L_1$ and $\nu(L_1, L_2) = L_2$.

The twistor correspondence is the set- valued mapping (or set- theoretic assignment).

$$\begin{cases} \tau: Z \in \mathbb{P} \longrightarrow \hat{Z} := \nu(\mu^{-1}(Z)) \subset \mathbb{M}, \\ \tau^{-1}: Z \in \mathbb{M} \longrightarrow L := \mu(\nu^{-1}(z)) \subset \mathbb{P}. \end{cases}$$

We recall that the basic diagram



is called the Penrose correspondence [4, 7].

Section 1.

In this section we use Batchelor Supermanifolds, [2]. Let $F = B^{4,n} - \{\omega \in B^{4,n} \mid \varepsilon(\omega) = 0\}$, where ε being augmentation map. F is an open set in coarse topology.

We define Sp $(3) = F/^{\sim}$ where the equivalence relation $\tilde{}$ is given by $\omega \sim \lambda\omega$ for any invertible element $\lambda \in B_0$, i.e, $\varepsilon(\lambda) \neq 0$, and it is called super projective twistor space.

The topology of F induces the quotient topology on Sp (3). Thus $SU \subset Sp(3)$ is open if $\Pi^{-1}(SU)$ is open in F, where $\Pi: F \longrightarrow Sp(3)$ is a projection, i.e, $\Pi(\omega) = [\omega]$.

Proposition 1. Sp (3) is a (3, n) dimensional supermanifold. And underlying body manifold of SP (3) is biholomorphic with the $P_3(\mathbb{C})$.

Proof. If
$SU_i = \{[b_1, b_2, b_3, b_4, \theta_1, ..., \theta_n] / [b_1, b_2, b_3, b_4, \theta_1, ..., \theta_n] \in Sp_1(3)$ and $\varepsilon(b_i) \neq 0\}$, $i \in \{1,2,3,4\}$. Then Sp (3) $= \bigcup_{i=1}^{4} SU_i$.

Every SU_i is an open set. i.e, $\Pi^{-1}(SU_i) = \varepsilon^{-1}(V_i)$ where V_i is an open set in $\mathbb{C}^4$. Indeed, $\Pi^{-1}(SU_i) = \{(x_1, x_2, x_3, x_4, \theta_1, ..., \theta_n) / \varepsilon(x_i) \neq 0\}$, $i \in \{1, 2, 3, 4\}$, now if $V_i = \{(C_1, C_2, C_3, C_4) \in \mathbb{C}^4 / C_i \neq 0\}$. Then V_i is subset of

F, and hence $\Pi^{-1}(SU_i) = \varepsilon^{-1}(V_i)$, $i \in \{1, 2, 3, 4\}$.

We define $S\varphi_i: SU_i \longrightarrow B^{3,n}$ such that

$S\varphi_1([b_1, b_2, b_3, b_4, \theta_1, \ldots, \theta_n])$
$= (b_2/b_1, b_3/b_1, b_4/b_1, \theta_1/b_1, \ldots, \theta_n/b_1)$
$S\varphi_2([b_1, b_2, b_3, b_4, \theta_1, \ldots, \theta_n])$
$= (b_1/b_2, b_3/b_2, b_4/b_2, \theta_1/b_1, \ldots, \theta_n/b_1)$,

and similarly $S\varphi_3$ and $S\varphi_4$.

And $S\varphi_2 o S\varphi_1^{-1}: S\varphi_1(SU_1 \cap SU_2) \longrightarrow S\varphi_2(SU_1 \cap SU_2)$ is defined by $S\varphi_2 o S\varphi_1^{-1}((b_2, b_3, b_4, \theta_1, \ldots, \theta_n)) = (1/b_2, b_3/b_2, b_4/b_2, \theta_1/b_2, \ldots, \theta_n/b_2)$ is supersmooth, and similarly for every invertable even element. Then every $(SU_i, S\varphi_i)$ is a chart and the set $\{(SU_i, S\varphi_i)\}^4_{i=1}$ is an atlas for the supermanifold Sp (3).

Suppose $X = Sp(3)/\sim$ where the equivalence relation "~" is defined by: p~q if and only if there exists a chart (SU, $S\varphi$) of Sp (3) such that p & q$\in$SU and $\varepsilon o S\varphi(p) = \varepsilon o S\varphi(q)$.

Then X has complex manifold structure and it is the underlying body manifold of SP (3), [1 & 5].

Structure X is $\{(U_i, \theta_i), i = 1,2,3,4\}$ where $U_i = \{[[p]] \mid [p] \in SU_i\}$ and $\theta_i: U_i \longrightarrow \mathbb{C}^3$ by $\theta_i([[p]]) = \varepsilon o S\varphi_i[p_1]$. And $\theta_2 o \theta_1^{-1}: \theta_1(U_1 \cap U_2) \longrightarrow \theta_2(U_1 \cap U_2)$ is defined by $\theta_2 o \theta_1^{-1}((C_1, C_2, C_3)) = (1/C_1, C_2/C_1, C_3/C_1)$ is holomorphism map and similarly $\theta_3 o \theta_2^{-1}$ and... .

Let $\{(W_i, \alpha_i)\}^4_{i=1}$ be the atlas for $P_3(\mathbb{C})$ where $W_i = \{[C_1, C_2, C_3, C_4] \mid C_i \neq 0\}$ then the function $f: X \longrightarrow P_3(X)$ defined by $f([[p]]) = [\varepsilon(p)]$ is well defined and is a biholomorphism, with inverse function $f^{-1}([C_1, \ldots, C_4]) = [[P]]$, where $[\varepsilon(P)] = [(C_1, \ldots, C_4)]$.■

Definition 1. Two members V and W of F are called linearly independent if for all α & $\beta \in B_0$ such that $\alpha V + \beta W$ is non-invertible, α & β are non-invertible, $\varepsilon(\alpha) = 0$ & $\varepsilon(\beta) = 0$.

Lemma 1. V & W are linear independent if and only if $\varepsilon(V)$ & $\varepsilon(W)$ are linear independent in $\mathbb{C}^4$.

Proof. Suppose V & W are linear independent. If

$$a\varepsilon(V) + b\varepsilon(W) = 0$$

Then we have

$$\varepsilon(\alpha)\varepsilon(V) + \varepsilon(\beta)\varepsilon(W) =$$
$$\varepsilon(\alpha V) + \varepsilon(\beta W) =$$
$$\varepsilon(\alpha V + \beta W) = 0,$$

where $a = \varepsilon(\alpha)$ and $b = \varepsilon(B)$ for some $\alpha, \beta \in B_0$.

Thus by linear independence of V & W we have $\varepsilon(\alpha) = 0$ and $\varepsilon(\beta) = 0$, i.e, a = b = 0, therefore $\varepsilon(V)$ and $\varepsilon(W)$ are linearly independent.

Conversely, suppose $\varepsilon(V)$ and $\varepsilon(W)$ are linearly independent and $\alpha V + \beta W$ is non-invertible, i.e, $\varepsilon(\alpha V + \beta W) = 0$ this implies $\varepsilon(\alpha)\varepsilon(V) + \varepsilon(\beta)\varepsilon(W) = 0$.

Then by linear independence of $\varepsilon(V)$ & $\varepsilon(W)$, we have $\varepsilon(\alpha) = 0$ and $\varepsilon(\beta) = 0$; that is, α and β are non-invertible. Therefore V and W are linearly independent by defintion 1.■

Definition 2. Let

$S_{v,w} = \{\alpha V + \beta W \mid \alpha \in B_0$ & $\beta \in B_0$, where V and W are linearly independent$\}$

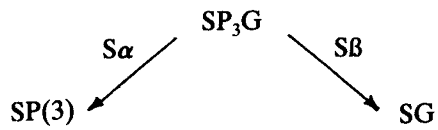
and $SG_1 = \{S_{v,w} \mid V \in F_0$ and $W \in F_0\}$, i.e, SG_1 is a set of subspaces of B_0^4 generated by two linearly independent vectors of F_0, that F_0 is even part of F.

We say that $SG = SG_1 \times B_1^n$ is a super Grassmannian manifold.

Suppose $SP_3G = \{(SL_1, SL_2) \mid SL_1$ is subspace of SL_2 where $SL_1 \in SP(3)$ & $SL_2 \in SG\}$.

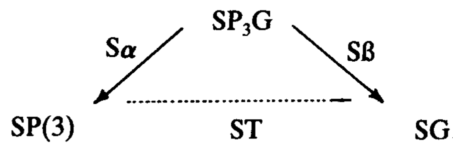
As in standard case, we can show like wise SG_1 & SP_3G are supermanifolds, that SP_3G is flag supermanifold.

Now we have the following natural diagram



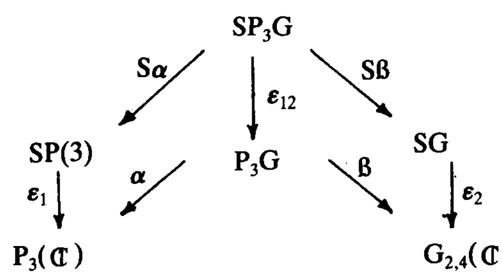
where $S\alpha(SL_1, SL_2) = SL_1$ and $S\beta(SL_1, SL_2) = SL_2$ are natural projections.

Using $S\alpha$ and $S\beta$ we can define a correspondence «ST» between SP (3) and SG, which is a set valued mapping,



Wedefine set valued ST by $ST(P) = S\beta(S\alpha^{-1}(P))$ and $ST^{-1}(q) = S\alpha(S\beta^{-1}(q))$. The above diagram is called super Penrose correspondence, and we call SP (3) a projective super twistor space and the set valued map ST, super twistor correspondence.

Theorem 1. If we apply an augmentation map ε to a super Penrose correspondence, we get a standard Penrose-correspondence. In other words, the following diagram commutes:



(*)

where $\varepsilon_1([P]) = [\varepsilon(P)]$, $\varepsilon_2(S_{v,w}, \theta_1, \ldots, \theta_n) := <\varepsilon(V), \varepsilon(W)>$, space generated by $\varepsilon(V)$ & $\varepsilon(W)$ and $\varepsilon_{12}((SL_1, SL_2)) = (\varepsilon_1(SL_1), \varepsilon_2(SL_2))$.

Proof. By proposition 1, ε_1 is defined by $\varepsilon_1([P]) = [\varepsilon(P)]$, for every $[P] \in SP(3)$.

By Lemma 1, the set of all two dimensional spaces generated by independent vectors $\varepsilon(V)$ & $\varepsilon(W)$ is $G_{2,4}(\mathbb{C})$, where $S_{v,w} \in SG_1$. $G_{2,4}(\mathbb{C})$ is Grassmanian manifold of a two dimensional vector space of $\mathbb{C}^4$.

Let $(SL_1, SL_2) \in SP_3G$, then $SL_1 = [P]$, for some $P \in F$ and $SL_2 = (S_{v,w}, \theta_1, \ldots, \theta_n)$, where V is even part of P, i.e., $P_0 = V$.

Since $\varepsilon_1(SL_1) = [\varepsilon(P)]$ and $\varepsilon_2(SL_2) = <\varepsilon(P), \varepsilon(W)>$, and we also know that $[\varepsilon(P)]$ is subspace of $<\varepsilon(P), \varepsilon(W)>$, therefore the set $\{(\varepsilon_1(SL_1), \varepsilon_2(SL_2)) / (SL_1, SL_2) \in SP_3G\}$ is P_3G, complex flag manifold.

Hence, when augmentation map is applied on SP(3), SG and SP_3G, we obtain $P_3(\mathbb{C})$, $G_{2,4}(\mathbb{C})$ and P_3G, that these spaces have the Penrose correspondence property.

Now we show that the diagram (*) commutes:

Let $(SL_1, SL_2) \in SP_3G$, then we have:

$\alpha o \varepsilon_{12}((SL_1, SL_2)) = \alpha((\varepsilon_1(SL_1), \varepsilon_2(SL_2)) = \varepsilon_1(SL_1)$,

on other hand, (1)

$\varepsilon_1 o S\alpha((SL_1, SL_2)) = \varepsilon_1(SL_1)$ (2)

(1) & (2) implies $\alpha o \varepsilon_1 = \varepsilon_1 o S\alpha$.

Also we have $\varepsilon_2 o S\beta((SL_1, SL_2)) = \varepsilon_2(SL_2)$, and

$\beta o \varepsilon_{12}((SL_1, SL_2)) = \beta(\varepsilon_1(SL_1, \varepsilon_2(SL_2)) = \varepsilon_2(SL_2))$, hence $\varepsilon_2 o S\beta = \beta o \varepsilon_{12}$. Therefore the diagram (*) commutes. ■

Further geometric properties of supertwistor's and super Penrose correspondence and comparing our approach with other approaches will be given elsewhere.

Section 2.

In this section we use Roger's definition of super smooth functions.

We denote by $F(B_0^3 \times B_1^n)$ the set of function $f(u, \zeta)$ over $B_0^3 \times B_1^n$, i.e, $f(u, \zeta): B_0^3 \times B_1^n \longrightarrow B$ where $u = (u_1, u_2, u_3, u_4)$ are even coordinates and $\zeta = (\zeta^1, \ldots, \zeta^n)$ are odd coordinates.

Every function $f(u, \zeta)$ is explained by

$$f(u, \zeta) = f_0(u) + \sum_{\alpha=1}^{n} \zeta^{\alpha} f_{\alpha}(u) + \ldots + \zeta^{\alpha_1} \zeta^{\alpha_2} \ldots \zeta^{\alpha_n} f_{\alpha_1 \alpha_2 \ldots \alpha_n}(u)$$

where $f_0(u) = \sum_{i_1, i_2, i_3, i_4} f_{i_1, i_2, i_3, i_4}(x)\, u_1^{i_1} u_2^{i_2} u_3^{i_3} u_4^{i_4}$ and $\varepsilon(u) = x$.

f_{α} and $f_{\alpha\beta}, \ldots$ are similarly found.

Definition 3. For every $i \in \{1, 2, 3, 4\}$ we define the derivative operator $D_i = \frac{\partial}{\partial u_i} + K \sum_{\alpha=1}^{n} \frac{\partial}{\partial \zeta} \alpha$, where K is an linear odd valued function.

Proposition 2. For every function $f(u, \zeta) \in F(B_0^3 \times B_1^n)$, $\varepsilon[D_j(f(u, \zeta))] = \frac{\partial}{\partial x_j}(\varepsilon o f(u, \zeta))$ where $\varepsilon(u_j) = x_j$, $j = 1, \ldots, 4$. In other words we have $\varepsilon o D_j = \frac{\partial}{\partial x_j} o \varepsilon$.

Proof.

$\varepsilon o D_j(f(u, \zeta))$

$= \varepsilon o D_j[f_0(u) + \sum_{\alpha=1}^{n} \xi^{\alpha} f_{\alpha}(u) + \ldots + \xi^{\alpha_1} \xi^{\alpha_2} \ldots \xi^{\alpha_n} f_{\alpha_1 \ldots \alpha_n}(u)]$

$= \varepsilon o D_j(f_0(u)) + \varepsilon o D_j[\sum_{\alpha=1}^{n} \xi^{\alpha} f_{\alpha}(u) + \ldots + \xi^{\alpha_1} \ldots \xi^{\alpha_n} f_{\alpha_1 \ldots \alpha_n}(u)]$.

However

$\varepsilon o D_j(f_0(u))$

$$= \varepsilon o D_j\Big(\sum_{k_1, k_2, k_3, k_4} f_{k_1, k_2, k_3, k_4}(X)\, u_1^{k_1} u_2^{k_2} u_3^{k_3} u_4^{k_4}\Big)]$$

$$= \varepsilon\Big[\frac{\partial}{\partial u_j}\Big(\sum_{k_1, k_2, k_3, k_4}^{n} f_{k_1 k_2 k_3 k_4}(X)\, u_1^{k_1} u_2^{k_2} u_3^{k_3} u_4^{k_4}\Big)\Big]$$

$$= \varepsilon\Big[\sum_{k_1, k_2, k_3, k_4}^{n} k_j f_{k_1 k_2 k_3 k_4}(X)\, u_1^{k_1} \ldots u_j^{k_j - 1} \ldots u_4^{k_4}\Big]$$

$$= \sum_{k_1, k_2, k_3, k_4}^{n} K_j f_{k_1 k_2 k_3 k_4}(x)\, x_1^{k_1} \ldots x_j^{k_j - 1} \ldots x_4^{k_4}. \quad (1)$$

We have

$$D_j\Big(\sum_{\alpha=1}^{n} \zeta^{\alpha} f_{\alpha}(u)\Big) = \sum_{\alpha=1}^{|n|} D_j(\zeta^{\alpha} f_{\alpha}(u))$$

$$= \sum_{\alpha=1}^{n}\Big[\Big(\frac{\partial}{\partial u_j} + K \sum_{\alpha=1}^{|n|} \partial/\partial\zeta^{\alpha}\Big)(\zeta^{\alpha} f_{\alpha}(u))\Big]$$

$$= \sum_{\alpha=1}^{n}\Big[\zeta^{\alpha} \frac{\partial}{\partial u_j} f_{\alpha}(u) + K()\, f_{\alpha}(u)\Big]$$

Hence $\varepsilon o D_j\left(\sum_{\alpha=1}^{n} \zeta^{\alpha} f_{\alpha}(u)\right) = 0$. We can similarly prove that $\varepsilon o D_j\left[\sum_{\alpha=1}^{n} \xi^{\alpha} f_{\alpha}(u) + \ldots + \xi^{\alpha_1} \ldots \xi^{\alpha_n} f_{\alpha_1 \ldots \alpha_n}(u)\right] = 0$. (2)

(1) & (2) imply

$\varepsilon o D_j[f(u, \zeta)) =$

$$\sum_{k_1, k_2, k_3, k_4}^{n} k_j f_{k_1 k_2 k_3 k_4}(X)\, X_1^{k_1} \ldots X_j^{k_j - 1} \ldots X_4^{k_4}. (*)$$

On the other hand we have

$$\frac{\partial}{\partial x_j}(\varepsilon o f(u,\zeta))$$
$$=\frac{\partial}{\partial x_j}\{\varepsilon f_0(u)+\varepsilon[\sum_{\alpha=1}^{n}\xi^\alpha f_\alpha(u)+\ldots+\xi^{\alpha_1}\xi^{\alpha_2}\ldots\xi^{\alpha_n} f_{\alpha_1..\alpha_n}(u)]\}$$
$$=\frac{\partial}{\partial x_j}(\varepsilon f_0(u)+0)$$

$$=\frac{\partial}{\partial x_j}(\varepsilon\sum_{k_1,k_2,k_3,k_4}^{n} f_{k_1k_2k_3k_4}(X)u_1^{k_1}..u_4^{k_4})$$

$$=\frac{\partial}{\partial x_j}(\sum_{k_1,k_2,k_3,k_4}^{n} f_{k_1k_2k_3k_4}(X)X_1^{k_1}..X_4^{k_4})$$

$$=\sum_{k_1,k_2,k_3,k_4}^{n} K_j f_{k_1k_2k_3k_4}(X)X_1^{k_1}\ldots X_j^{k_j-1}..X_4^{k_4}.(**)$$

Therefore (*) is equal to (**), that is, $\varepsilon o D_j=\frac{\partial}{\partial x_j} o\varepsilon$ for $j\in\{1,2,3,4\}$.■

Lemma 2. If $f(u,\zeta)$ is noninvertible, then for every (ω,θ) such that $\varepsilon(u,\zeta)=\varepsilon(\omega,\theta)$, $f(\omega,\theta)$ is also nonivertible.

Proof. $f(u,\zeta)$ is non-invertible, i.e, $\varepsilon f(u,\zeta)=0$. But
$\varepsilon o f(u,\zeta)$
$$=\varepsilon[f_0(u)+\sum_{\alpha=1}^{n}\xi^\alpha f_\alpha(u)+\ldots+\xi^{\alpha_1}\xi^{\alpha_2}\ldots\xi^{\alpha_n} f_{\alpha_1\alpha_2\ldots\alpha_n}(u)]$$
$$=\varepsilon o f_0(u)$$
$$=\varepsilon\sum_{k_1,\ldots,k_l} g_{k_1k_2..k_l}(X)u_1^{k_1}..u_l^{k_l})\text{ where }\varepsilon(u)=x$$

$$=\sum_{k_1,\ldots,k_l} g_{k_1k_2..k_l}(X)\varepsilon(u_1)^{k_1}..\varepsilon(u_l)^{k_l}$$

$$=\sum_{k_1..k_l} g_{k_1k_2..k_l}(X)\varepsilon(\omega_1)^{k_1}..\varepsilon(\omega_k)^{k_l}[(\varepsilon(u)-\varepsilon(\omega))]$$
$$=\varepsilon f_0(\omega)+0$$
$$=\varepsilon f_0(\omega)+\varepsilon\sum\xi^\alpha f_\alpha(\omega)+\ldots$$
$$=\varepsilon o f(\omega,\theta).$$

Hence $\varepsilon o f(\omega,\theta)=0$. We conclude that $f(\omega,\theta)$ is noninvertible.■

Definition 4. Let $A=\{(\omega,\theta)\,/\,g(\omega,\theta)$ is nonivertible$\}$. If for every pair $(\omega_1,\theta_1)\in A$ and $(\omega_2,\theta_2)\in A$ we have $\varepsilon(\omega_1,\theta_1)=\varepsilon(\omega_2,\theta_2)$, then A is called the singularity set of function $h(u,\zeta)=f(u,\zeta)/g(u,\zeta)$.

Definition 5. The singularity set $A\subset\Omega$, Ω is an open set, is called an m-fold pole set of function F if

i) F is analytic (i.e, has a power series expansion) over Ω-A

ii) There exist $b_1,\ldots,b_m\in B$ such that $F(p)-\sum_{i=1}^{m}\frac{b_i}{(P-a)^i}$ is analytic for every $a\in A$.

Proposition 3. Suppose f has an m-fold pole set in ε^{-1} (b), then $\varepsilon o f$ has a pole of order m at the point b.

Proof. Suppose $g(p)=f(p)-\sum_{i=1}^{m}\frac{b_i}{(P-a)^i}$ where $a\in\varepsilon^{-1}$ (b). By definition 5, g(p) has a series expansion and it is analytic. Therefore $\varepsilon g(p)=g_0(\varepsilon(p))$ is analytic.

Since b is a singularity point of $\varepsilon o f$ and $\varepsilon g(p)=\varepsilon o f(p)-\sum_{i=1}^{m}\frac{\varepsilon b_i}{(\varepsilon(P-a))^i}$ is analytic, therefore b is pole of order m of the function $\varepsilon o f$.■

In this paper, we use Roger's definition of «contour integrals» in a one-dimensional even superspace, [6].

Lemma 3. $\varepsilon\oint_\gamma du/u=2\pi i$ where γ is a closed path surrounding the singularity set of 1/u. u is an even coordinate.

Proof. $$\int_\gamma du/u=\int_0^{2\pi}\frac{\gamma'(t)}{\gamma(t)}dt=\int_0^{2\pi}\frac{B'(t)+S'(t)}{B(t)+S(t)}dt$$

$$=\oint_0^{2\pi}\frac{B'(t)dt}{B(t)+S(t)}+\oint_0^{2\pi}\frac{S'(t)}{B(t)+S(t)}dt$$

where $\gamma(t)=B(t)+S(t)$.

Here B(t) is the body and S(t) is soul of $\gamma(t)$. Since $B(t)+S(t)$ is invertible, it has an inverse; i.e, there exists $\frac{1}{B(t)}+C(t)$ such that $(B(t)=S(t))(\frac{1}{B(t)})+C(t)=1$. Therefore

$$\varepsilon\oint_\gamma du/u=\varepsilon\int_0^{2\pi}\frac{B'(t)dt}{B(t)+S(t)}=\varepsilon\int_0^{2\pi}(\frac{1}{B(t)}+C(t))B'(t)\,dt$$

$$=\varepsilon\int_0^{2\pi}\frac{B'(t)}{B(t)}dt+\varepsilon\int_0^{2\pi}C(t)B'(t)\,dt=\varepsilon(2\pi i)+0=2\pi i.$$■

Proposition 4. **Let A_j, $j=1,2,\ldots,m$ be singularity sets of f and γ be a closed path such that f is analytic over γ and γ surrounds singularity sets.**
$\varepsilon\oint_\gamma f(u)\,du=\oint_{\varepsilon o\gamma}\varepsilon o f(u)\,dx$ where $a_j\in A_j$ and $b_j\in B_0$.

Proof. By definition g(u) is an analytic function. Hence by the Generalised Cauchy theorem [6] we have $\oint_\gamma g(u)du=0$.

This implies that $\oint_\gamma f(u)\,du=\sum_{j=1}^{m}\oint_\gamma \frac{b_j}{u-a_j}\,du$

$$=\sum_{j=1}^{m} b_j \oint_\gamma \frac{du}{u-a_j}.$$

But by Lemma 4 we have

$$\varepsilon\oint_\gamma \frac{du}{u-a_j}=\varepsilon\oint_0^{2\pi}\frac{\gamma'(t)dt}{\gamma(t)-a_j}=\varepsilon\int_0^{2\pi}\frac{\Gamma'(t)}{\Gamma(t)}dt=2\pi i,$$

where $\gamma(t)-a_j=\Gamma(t)$. Therefore

$$\varepsilon\oint_\gamma f(u)du=\varepsilon\sum_{j=1}^{m} b_j\oint_\gamma\frac{du}{u-a_j}$$
$$=\sum_{j=1}^{m}\varepsilon(b_j)(2\pi i)$$
$$=2\pi i\sum_{j=1}^{m}\varepsilon(b_j). \qquad (1)$$

On the other hand, we know that $\varepsilon f(u)$ has a finite number of singularity points $\varepsilon(a_j)$, $j=1,\ldots,m$, and $\varepsilon f(u)$ $-\sum_{j=1}^{m}\frac{\varepsilon(b_j)}{\varepsilon(u)-\varepsilon(a_j)}=\varepsilon g(u)$ is analytic. Hence $\oint_{\varepsilon o\gamma}\varepsilon o f(u)\,d(\varepsilon(u))=\oint_{\varepsilon o\gamma}\sum_{j=1}^{m}\frac{\varepsilon(b_j)}{\varepsilon(u)-\varepsilon(a_j)}dx$

$$=\sum_{j=1}^{m}\varepsilon(b_j)\oint\frac{dx}{\varepsilon(u)-\varepsilon(a_j)}=2\pi i\sum_{j=1}^{m}\varepsilon(b_j). \qquad (2)$$

(1) & (2) imply $\oint_\gamma f(u)\,du=\oint_{\varepsilon o\gamma}\varepsilon f(u)dx$. ■

Lemma 4. If γ be a closed path surrounding the singularity set of $\frac{1}{u^n}$, $n\geq 2$, then $\varepsilon\oint_\gamma\frac{du}{u^n}=0$.

Proof. We have

$$\varepsilon\oint_\gamma\frac{du}{u^n}=\varepsilon\int_0^{2\pi}\frac{\gamma'(t)}{(\gamma(t))^n}dt$$
$$=\varepsilon\int_0^{2\pi}\frac{\beta'(t)+S'(t)}{[\beta(t)+S(t)]^n}dt$$
$$=\varepsilon\int_0^{2\pi}\frac{\beta'(t)dt}{[\beta(t)+S(t)]^n}$$
$$=\varepsilon\int_0^{2\pi}\beta'(t)\left(\frac{1}{[\beta(t)]^n}+C(t)\right)dt$$
$$=\varepsilon\int_0^{2\pi}\frac{\beta'(t)dt}{[\beta(t)]^n}=\varepsilon(0)=0$$

Where $\gamma(t)=\beta(t)+S(t)$, $\beta(t)$ is the body and $S(t)$ is the soul of $\gamma(t)$ and $\left(\frac{1}{[\beta(t)]^n}+C(t)\right)$ is the inverse of $(\beta(t)+S(t))^n$. ■

Proposition 5. Let the even function f have an m-fold pole set $A=\varepsilon^{-1}(b)$, then

$$\varepsilon\oint_\gamma f(u)du=\int_{\varepsilon o\gamma}\varepsilon o f(u)\,dx$$

Where γ is a closed path surrounding A.

Proof. By definition 5, the function $g(u)=f(u)-\sum_{k=1}^{m}\frac{(b_k)}{(u-a)^k}$ is an analytic function for every $a\in A$, and by Lemma 3 and 4 we have:

$$\varepsilon\oint_\gamma f(u)du=\sum_{k=1}^{m}\varepsilon(b_k)\,\varepsilon\oint_\gamma\frac{du}{(u-a)^k}=$$
$$\varepsilon(b_1)\,\varepsilon\oint_\gamma\frac{du}{u-a}+\sum_{k=2}^{m}\varepsilon(b_k)\frac{du}{(u-a)^k}$$
$$2\pi i\,(\varepsilon(b_1))+0=2\pi i\,(\varepsilon(b_1)). \qquad (1)$$

On the other hand, we know by proposition 3 that $\varepsilon o f(u)$ has a pole b of order m, thus

$$\varepsilon o f(u)-\sum_{k=1}^{m}\frac{\varepsilon(b_k)}{(\varepsilon(u)-b)^k}=\varepsilon o g(u)\text{ is analytic.}$$

Hence $\oint_{\varepsilon o\gamma}\varepsilon o g(u)dx=0$, where $x=\varepsilon(u)$. Therefore

$$\oint_{\varepsilon o\gamma}\varepsilon o f(u)dx=\sum_{k=1}^{m}\varepsilon(b_k)\oint_{\varepsilon o\gamma}\frac{dx}{(\varepsilon(u)-b)^k}$$
$$=\varepsilon(b_1)\oint_{\varepsilon o\gamma}\frac{dx}{(x-b)}+\sum_{k=2}^{m}\varepsilon(b_k)\oint_{\varepsilon o\gamma}\frac{dx}{(x-b)^k}$$
$$=\varepsilon(b_1)(2\pi i). \qquad (2)$$

(1) and (2) imply $\varepsilon\oint_\gamma f(u)\,du=\int_{\varepsilon o\gamma}\varepsilon o f(u)d(\varepsilon(u))$. ■

Section 3.

In this section, we use section 2 to introduce Super-Zero-Rest-Mass equations, a solution of which will be represented by contour integral.

Let $\tilde{\nabla}^{AA'}=\frac{\partial}{\partial u_{AA'}}+K\sum_{\alpha=1}^{N}\left(\frac{\partial}{\partial\theta_{\alpha A}}+\frac{\partial}{\partial\theta_{\alpha A'}}\right)$, where

A and $A'\in\{0,1\}$ and K is an linear odd valued function.

If

$$S\varphi_{B...D} = \varphi_{B...D} + \theta_{\alpha A}\,\varphi^{\alpha A}{}_{B...D} + \theta_{\alpha A}\,\theta_{\beta A}\,\varphi^{\alpha\beta A}{}_{BC...D} + \ldots +$$

$$\theta_{1A}\,\theta_{2A}\ldots\theta_{nA}\,\varphi^{1}{}_{B}{}^{2}{}_{C}\ldots{}^{n}{}_{D}{}^{A}$$

and

$$S\varphi_{B'...D'} = \varphi_{B'...D'} + \theta_{\alpha A'}\,\varphi^{\alpha A'}{}_{B'...D'} + \ldots +$$

$$\theta_{1A'}\,\theta_{2A'}\ldots\theta_{nA'}\,\varphi^{12...nA}_{B'C'...D'}$$

(Summation Convention assumed)

where $\varphi_{B...D}$ & $\varphi^{\alpha A}{}_{B...D}$ & ... are standard symmetric spinor fields and $\alpha, \beta, \ldots, \in \{1, \ldots, n\}$ and other index are 0, 1. We say $S\varphi_{AB...D}$ and $S\varphi_{A'B'...D'}$ are super spinor fields.

Definition 6. Equations $\tilde{\nabla}^{AA'} S\varphi_{AB...D} = 0$ and $\tilde{\nabla}^{AA'} S\varphi_{A'B'...D'} = 0$ (summation convention assumed.) are called Super- Zero- Rest- Mass equations, (S. Z. R. M).

By proposition 2 we have $\varepsilon\,\tilde{\nabla} = \tilde{\nabla}\varepsilon$. Therefore if we apply an augumentation map ε to Super- Zero- Rest- mass equations, we get standard Zero- Rest- mass equations.

Also we have the following theorem:

Theorem 2. Let f be an analytic, choose a non- negative integer 2s and for r= 0, 1,..., 2s put for every $\alpha \in \{1, \ldots, n\}$ $S\varphi_r =$

$$\frac{1}{2\pi i}\oint \lambda^r f(\lambda, u+\lambda y, w+\lambda v, \theta_{\alpha_0} - \theta_{\alpha_0'}, \theta_{\alpha_1} - \theta_{\alpha_1'}, \theta_{\alpha_1} - \theta_{\alpha_0'}, \theta_{\alpha_0} - \theta_{\alpha_1'})\,d\lambda$$

where $\lambda \varepsilon B_o$ and u, y, w, v are even coordinates and θ_α - θ_α , are odd coordinates. The contour Surrounds Singularities Sets of f and varies continuously with u, v, w, y, θ_{α_0}, $\theta_{\alpha_{1'}}$, $\theta_{\alpha_{0'}}$. Then $S\varphi_r$ is solution of the Super- Zero- Rest- Mass equations and $\varepsilon(S\varphi_r)$ is solution of the standard Zero- Rest- Mass equations.

Proof. We have

$$\left[\frac{\partial}{\partial y} + K\sum_{\alpha=1}^{n}\left(\frac{\partial}{\partial\theta_{\alpha_A}} + \frac{\partial}{\partial\theta_{\alpha_{A'}}}\right)\right](S\varphi_r) = \left[\frac{\partial}{\partial u} + K\sum_{\alpha=1}^{n}\left(\frac{\partial}{\partial\theta_{\alpha_A}} + \frac{\partial}{\partial\theta_{\alpha_A}}\right)\right](S\varphi_{r+1}), (1)$$

and

$$\left[\frac{\partial}{\partial v} + K\sum_{\alpha=1}^{n}\left(\frac{\partial}{\partial\theta_{\alpha_A}} + \frac{\partial}{\partial\theta_{\alpha_A}}\right)\right](S\varphi_r) = \left[\frac{\partial}{\partial w} + K\sum_{\alpha=1}^{n}\left(\frac{\partial}{\partial\theta_{\alpha_A}} + \frac{\partial}{\partial\theta_{\alpha_{A'}}}\right)\right](S\varphi_{r+1}), (2)$$

r=0, 1,..., 2s-1 and A & A′ $\in \{0, 1\}$.

If we put $S\varphi_0 = S\varphi_{000...0}$, $S\varphi_1 = S\varphi_{100...0}$, $S\varphi_2 = S\varphi_{110...0}$, ... , $S\varphi_{2s} = S\varphi_{111...1}$ where $\varphi_{ABC...K}$ has 2s indices and is symmetric: $S\varphi_{ABC...K} = S\varphi_{(ABC...K)}$, and the spinor notation $u_{00'} = v$, $u_{01'} = -y$, $u_{10'} = -w$ and $u_{11'} = u$ are used, then we have:

$$\tilde{\nabla}^{A0'} S\varphi_{ABC...D} = \tilde{\nabla}^{00'} S\varphi_{0BC...D} + \tilde{\nabla}^{10'} S\varphi_{1BC...D} =$$

$$\left[\frac{\partial}{\partial u_{00'}} + K\sum_{\alpha=1}^{n}\left[\frac{\partial}{\partial\theta_{\alpha 0}} + \frac{\partial}{\partial\theta_{\alpha 0}}\right]\right][S\varphi_{0BC...D}] +$$

$$\left[\frac{\partial}{\partial u_{10'}} + K\sum_{\alpha=1}^{n}\left[\frac{\partial}{\partial\theta_{\alpha 1}} + \frac{\partial}{\partial\theta_{\alpha 0'}}\right]\right][S\varphi_{1BC...D}] =$$

$$\left[\frac{\partial}{\partial v} = K\sum_{\alpha=1}^{n}\left[\frac{\partial}{\partial\theta_{\alpha 1}} + \frac{\partial}{\partial\theta_{\alpha 0'}}\right]\right][S\varphi_r] +$$

$$\left[-\frac{\partial}{\partial W} + K\sum_{\alpha=1}^{n}\left[\frac{\partial}{\partial\theta_{\alpha 1}} + \frac{\partial}{\partial\theta_{\alpha 0'}}\right]\right][S\varphi_{r+1}] = 0, \text{ by } (1)$$

And also $\tilde{\nabla}^{A1'} S\varphi_{ABC...D} = 0$, by (2).

Therefore $\tilde{\nabla}^{AA'} S\varphi_{ABC...D} = 0$, this implies that $S\varphi_r = \frac{1}{2\pi i}\oint \lambda^r f(\lambda, u+\lambda y, w+\lambda v, \theta_{\alpha_0} - \theta_{\alpha_{0'}}, \ldots)\,d\lambda$ is solution of the (S. Z. R. M) equations.

By proposition 5 we have

$$\varepsilon S\varphi_r = \frac{1}{2\pi i}\oint (\varepsilon\lambda)^r \varepsilon o f(\lambda, u+\lambda y, w+\lambda v, \theta_{\alpha_0} - \theta_{\alpha_{0'}}, \ldots)\,db$$

$$= \frac{1}{2\pi i}\oint b^r f_0(b, x_1 + \lambda x_2, x_4 + \lambda x_3)\,db,$$ where $\varepsilon(\lambda) = b$, $\varepsilon(u) = X_1$, $\varepsilon(y) = x_2$, $\varepsilon(v) = x_3$ and $\varepsilon(w) = x_4$, and f_0 is the body of function of f. If $\varepsilon S\varphi_r = \varphi_r$, then we have seen that φ_r is solution of the Zero- Rest- Mass equations [3].■

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